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AUTHOR Dickinson, Jonathan
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ABSTRACT

The study of labor supply is directed to a theoretical methodology under which the choice of the general functional form of the income-leisure preference structure may be regarded as an empirical question. The author has reviewed the common functional forms employed in empirical labor supply models and has characterized the inherent preference structures in terms of patterns of expansion paths in the plane of leisure and market consumption. The preference maps for the more tractable models are seen to fall within a well-behaved set of alternative structures that range along a continuum from divergent to convergent patterns of expansion paths. A new class of preference structures, having parallel expansion paths, falls within this continuum. A proposed empirical model for the estimation of the form of the preference structure is adapted from an estimation form developed by Cohen, Rea, and Lerman (1970). This model permits the estimation of sample analogs of expansion paths at different wage rates. The pattern of the estimated path may then be tested against those implied by alternative analytical functions. (Author/EA)

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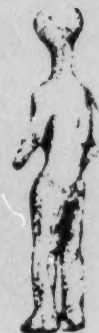


IMPLICIT AND EXPLICIT PREFERENCE
STRUCTURES IN MODELS OF LABOR SUPPLY

Jonathan Dickinson

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
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Abstract

In the economic theory of labor supply, a worker's labor supply behavior is presumed to be determined by an underlying structure of preferences for leisure and market consumption. In most empirical studies of labor supply, the functional form of the inherent preference structure has been chosen by a priori assumption, either implicitly through the choice of a form for the empirical labor supply function or explicitly through the choice of a specific income-leisure utility function. In this paper we develop a theoretical methodology under which the choice of the general functional form of the income-leisure preference structure may be regarded as an empirical question. We review the common functional forms employed in empirical labor supply models and characterize the inherent preference structures in terms of patterns of expansion paths in the plane of leisure and market consumption. The preference maps for the more tractable models are seen to fall within a well-behaved set of alternative structures that range along a continuum from divergent to convergent patterns of expansion paths. A new class of preference structures, having parallel expansion paths, falls within this continuum. A proposed empirical model for the estimation of the form of the preference structure is adapted from an estimation form developed by Cohen, Rea, and Lerman (1970). This model permits the estimation of sample analogs of expansion paths at different wage rates. The pattern of the estimated path may then be tested against those implied by alternative analytical functions.

IMPLICIT AND EXPLICIT PREFERENCE STRUCTURES IN MODELS OF LABOR SUPPLY

I. Introduction

The theoretical models for most recent empirical studies of labor supply have been cast in the mold of classical demand analysis. In these models, an individual worker-consumer is presumed to have a stable structure of preferences for leisure and market consumption that may be represented by a well-behaved utility function of those two basic goods. Given an opportunity set, defined by a wage rate and a level of nonwage income, an individual is expected to choose the combination of leisure and market goods that maximizes his or her utility. Optimal choices of leisure and market consumption over a range of wage rates and nonwage income may then be represented by a set of demand functions for the two goods. The labor supply function follows directly from the demand function for leisure, since labor supply is conceptualized as the simple additive complement of leisure demand.

In the classical general model of labor supply, the only assumptions made about the utility function concern general convexity and continuity properties. As a consequence, the general theory provides very little information about the form of the labor supply function. The theoretical implications are limited to a familiar set of restrictions on the partial derivatives of the supply function evaluated at a given equilibrium point. The fully general model implies little about responses to large changes in budget variables or about the interrelationships among labor supply parameters evaluated

at different equilibrium points. These kinds of information cannot be obtained without more specific knowledge about the structure of preferences for leisure and market goods over a broad range of those variables.

Such knowledge is critically necessary for many of the practical applications of labor supply models. The problem of simulating labor supply responses to proposed income-maintenance programs is a case in point. A typical negative income tax proposal would provide income supplements of several thousand dollars combined with 50 percent cuts in workers' net marginal wage rates. Point estimates of labor supply parameters at a worker's preprogram equilibrium point are clearly not sufficient to simulate responses to so large a change. Accurate simulation also requires information about changes in labor supply parameters over the relevant range of variables. Most empirical studies of labor supply have been based on a priori assumptions about the broader structure of the labor supply function or of the underlying preference function. There has been very little empirical testing of the restrictions inherent in the various assumptions about functional form. As a consequence, inferences about labor supply responses to major changes in economic incentives may be subject to distortion due to inappropriate functional assumptions.

Two distinct approaches to the structure of income-leisure preferences¹ are evident in recent empirical labor supply literature. We shall refer to these two approaches as classical general models and explicit utility models. In studies based on the classical

general model, the question of the broad structure of preferences has largely been ignored. The primary emphasis has been placed on estimating derivatives of the labor supply function and testing those estimates for theoretical acceptability at a single equilibrium point. In explicit utility models, a variety of specific functional forms has been assumed for the income-leisure utility function. The parameters of the derived labor supply model explicitly correspond to the parameters of the preference structure. Estimates of the labor supply parameters thus imply estimates of the full preference structure, subject to the limitations of the assumed functional form.

Explicit utility models have differed in the level of generality of the assumed utility function. Clearly, those studies based on more general utility functions are less likely to produce distorted estimates of the preference structure arising from limitations of the assumed functional form. In general, however, little attention has been paid to testing the appropriateness of the assumed functional form. Studies based on the classical general model have also differed in their true generality. The majority of these studies have assumed explicit functional forms for the supply function in order to estimate the desired parameters. Assumptions about the supply function imply corresponding assumptions about the underlying preference structure that are often no less restrictive than the assumptions of explicit utility models. Again, little attention has been paid to testing of the appropriateness of these restrictions.

This paper is part of a larger project² to obtain estimates of income-leisure preference structures that are not limited by arbitrary assumptions about functional form. A large part of the paper is

devoted to a review of analytical labor supply models, with particular emphasis on the properties of the underlying preference structures. A theoretical methodology is developed under which preference structures are characterized by patterns of expansion paths at different marginal wage rates. This methodology facilitates comparisons between the structures that are implicit in the various adaptations of the classical general labor supply model and the structures that are assumed in explicit utility models.

The review of preference structures inherent in common analytical models provides the theoretical background for interpretation of parameter estimates from a general empirical model. Our proposed model is designed to allow estimation of sample analogs of expansion paths subject to only minimal functional restrictions. The estimation form is adapted from a flexible model developed by Cohen, Rea, and Lerman (1970) and elaborated by Rea (1971). The expansion path methodology provides the framework for an integrated interpretation of the matrix of isolated point estimates from that model and facilitates general inferences about the underlying preference structure.

The discussion is in two major parts corresponding to the contrasting approaches to labor supply models noted above. Section II begins with a brief review of the fully general classical demand theory model and then turns to a discussion of specific functional approximations that have been employed for the estimation of labor supply parameters. The models reviewed include the linear additive model, nonlinear additive models, double logarithmic models, and the

model suggested by Ashenfelter and Heckman. For the more tractable forms we derive a full map of the preference structure in terms of expansion paths. For others, we consider only selected features of the implicit preference structure and evaluate their theoretical plausibility. The section concludes with an exposition of the Cohen, Rea, and Lerman model and of our proposed interpretation of estimates from that model.

In Section III we review the preference structures implied by a variety of explicit utility models. These models include the Stone-Geary linear expenditure model, constant elasticity of substitution (C.E.S.) models with subsistence parameters, variable elasticity models, and the quadratic utility model. In addition to reviewing the utility models employed in previous empirical work, we discuss the properties of a new class of preference structures characterized by parallel expansion paths.

Throughout the paper we restrict our attention to labor supply models for one-worker families. We also abstract fully from any stochastic elements of labor supply and from any empirical problems that may affect estimates of labor supply parameters.³ The idealized world of stable, nonstochastic labor supply functions serves best for our analysis of the implicit and explicit preference structures of analytical labor supply models.

II. Classical General Labor Supply Models

A. The Fully General Model

The fully general model of labor supply based on classical demand theory embodies very few assumptions about the structure of

preferences for leisure and market consumption. It is assumed that a worker's preferences for the two basic goods may be represented by a utility function as in (1), with leisure denoted by L and market consumption by M .

$$(1) \quad u = u(L, M) \quad .$$

The utility function (1) is assumed to be increasing, twice differentiable, and convex, but no assumptions are made about its specific functional form. A worker is presumed to choose the levels of leisure and market consumption that maximize his or her utility subject to the budget constraint (2).

$$(2) \quad a) \quad M = w(T-L) + I$$

or

$$b) \quad M = wH + I$$

where

w is the worker's wage rate in real terms,

T is the total time available,

$H=(T-L)$ is the time spent at market work, and

I is the real value of nonwage income.

For notational simplicity, the price of market goods⁴ has been set equal to unity and wage rates and income are expressed in real terms.

For any well-behaved utility function, the quantities of leisure and market goods that maximize utility subject to the budget constraint will be functions of nonwage income and the wage rate.

$$(3) \quad a) \quad L = L(w, I) \quad ;$$

$$b) \quad M = M(w, I) \quad .$$

The demand functions (3a, 3b) are as general as the utility function. In the absence of additional assumptions or empirical estimates, we have no information about their specific functional form. The demand functions do satisfy the familiar restrictions that follow from the general characteristics of the preference function and of maximizing behavior. It will be more convenient to express these restrictions in terms of the labor supply function, since labor supply is our primary interest.⁵

The labor supply function follows directly from the leisure demand equation (3a) by means of the work-leisure identity.

$$(4) \quad a) \quad H = T - L(w, I) ;$$

$$b) \quad H = H(w, I) .$$

The total time available, T , is taken to be a constant, so the properties of the labor supply function (4b) are equivalent to those of the demand function for leisure except for sign changes.

The demand function for market goods may then be expressed in terms of the labor supply function by means of the budget constraint.⁶

$$(5) \quad M(w, I) = w \cdot H(w, I) + I .$$

In the following discussion we will focus primarily on the labor supply function. However, it is important to keep in mind that market consumption is an endogenous variable to be determined jointly with labor supply. On occasion we will refer to market consumption, M , as "total family income" or in the adjectival phrase "income-leisure." For the most part we reserve the term "income," and especially "income effect," to refer to nonwage income, I , which is presumed to be exogenous.

The central features of the general labor supply model are contained in the Slutsky equation and the restrictions on its parameters.

$$(6) \quad dH = (S + H \frac{\partial H}{\partial I})dw + \frac{\partial H}{\partial I} dI$$

The parameter, S , is the pure substitution effect. It represents the response of an individual's work hours to an infinitesimal change in the wage rate if nonwage income is also changed so that the level of utility is held constant. Equivalently, the substitution effect is the response of work hours to an infinitesimal change in the marginal wage with the average wage unchanged.⁷

The central theoretical restriction on the parameters of the Slutsky equation is that the substitution effect must be positive or zero⁸:

$$(7) \quad S > 0 .$$

Usually the pure substitution effect is not directly observable. The parameters that can be measured directly are the labor supply responses to nonwage income and to changes in wage rates for all hours of work. Each response is measured with other variables held constant or controlled in a multivariate statistical model. The relationships of these observable responses to the pure substitution effect follow from equation (6).

$$(8) \quad \frac{\partial H}{\partial w} = S + H \frac{\partial H}{\partial I} .$$

The restriction on observable parameters is thus given by equation (9).

$$(9) \quad S = \frac{\partial H}{\partial w} - H \frac{\partial H}{\partial I} \geq 0 .$$

Restrictions on the signs of the income effects do not follow from the strictly general model, but it is generally assumed that leisure is a normal good. Under that assumption the income effect on leisure is positive and the income effect on leisure is negative.

$$(10) \quad a) \quad \frac{\partial L}{\partial I} > 0 ;$$

$$b) \quad \frac{\partial H}{\partial I} < 0 .$$

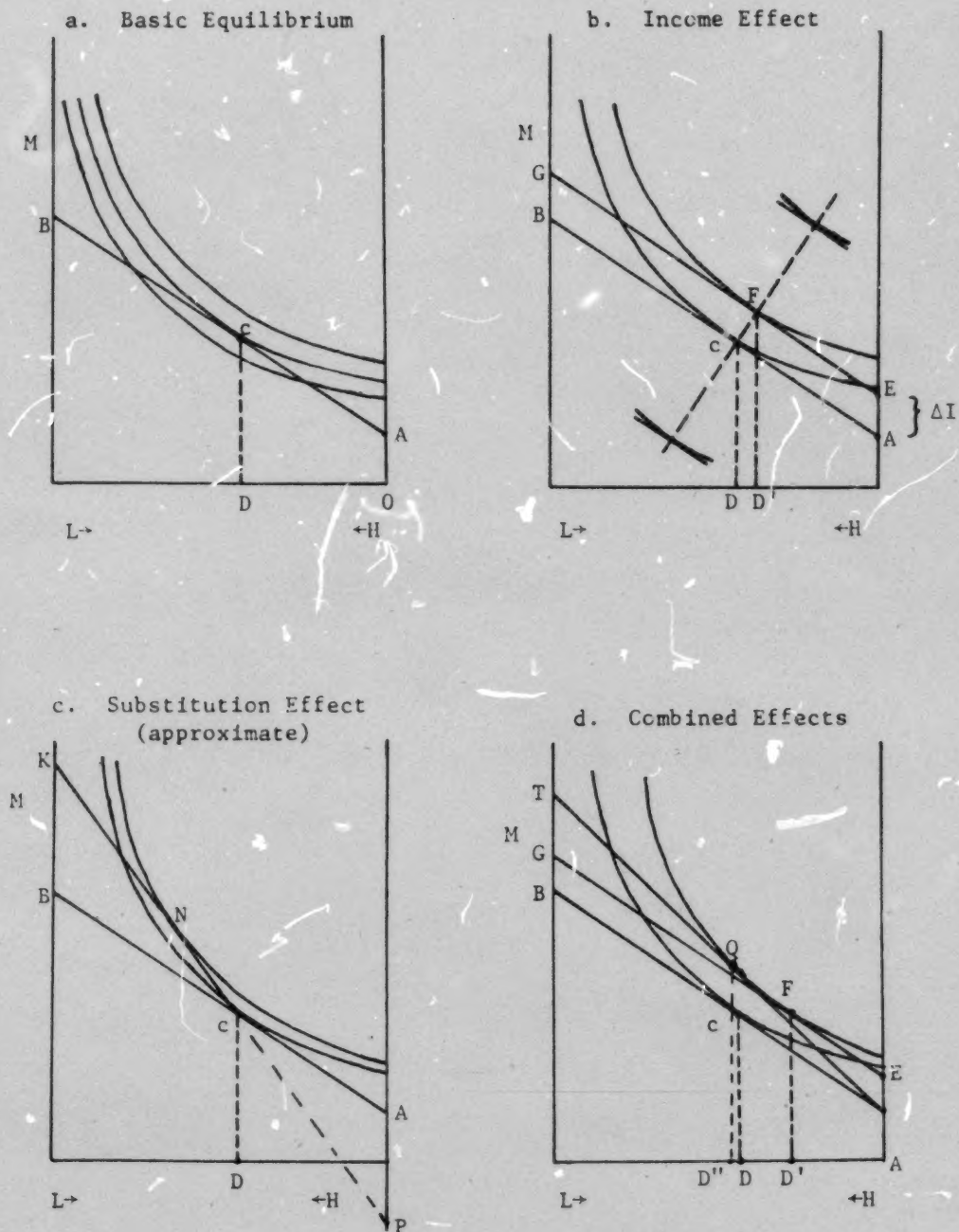
Market consumption is also assumed to be a normal good. Given our expression of demand for market goods in terms of the labor supply function, this assumption translates into another restriction on the income effect in the labor supply function.

$$(11) \quad a) \quad \frac{\partial M}{\partial I} = w \frac{\partial H}{\partial I} + 1 > 0 ;$$

$$b) \quad - \frac{\partial H}{\partial I} < \frac{1}{w} .$$

The decomposition of the observable wage response into income and substitution effects is particularly amenable to graphical presentation. This graphical presentation also serves as a useful basis for our subsequent discussion of the implications of empirical labor supply functions. For the simple model with only one worker, the utility function and the opportunity set can be mapped on a plane with market goods and the leisure of the worker as axes. A portion of the utility map is shown in Figure 1a, along with an initial wage-income opportunity set AB and the resulting optimal equilibrium labor supply position at point C. Figures 1b-1d

Figure 1. Graphical Representation of Income and Substitution Effects in the Simple Labor Supply Model



represent changes in the optimal labor supply equilibrium in response to changes in the income and wage-rate variables that define the opportunity set.

In Figure 1b, nonwage income has been increased by an amount ΔI , and the new opportunity set is given by line EG, with the new optimum of labor supply and market consumption at point F. Given the assumption that leisure is a normal good, the additional nonwage income results in a decrease in equilibrium work hours, as shown. The decrease in work hours per unit of nonwage income is referred to as the income effect on work hours or simply the income effect. The income effect on market consumption is illustrated by the vertical distance between points C and F. That parameter is positive if market consumption is a normal good but is less than unity because a portion of the change in nonwage income is allocated to the purchase of leisure.

The dashed line through points C and F in Figure 1b is an expansion path, the locus of equilibrium values of labor supply and market goods that result from changing nonwage income while holding wage rate constant. The illustrated expansion path is linear, though this need not be generally true. Other expansion paths will be defined for other given values of the wage rate and generally may differ in slope and/or curvature. In anticipation of subsequent discussions in this section, note that a set of expansion paths for a variety of wage rates embodies much of the same information about a preference map as does a set of indifference curves.

An approximation of the substitution effect is illustrated in Figure 1c. From an original equilibrium point C, a worker is offered an increased marginal wage rate represented by the line CK.⁹ The new equilibrium point N illustrates the positive substitution effect on work hours of a higher marginal wage. Because the figure shows a finite change in the marginal wage, the new equilibrium is at a higher level of utility. In the limit of successive infinitesimal changes in the marginal wage rate, the new equilibrium will fall arbitrarily close to the original indifference curve.

The combination of effects resulting from a change in wage rate for all hours worked is shown in Figure 1d. If work hours were unchanged, an increase in wage rate would result in an increase in total money income equal to $H_0 \Delta w$. If this increase came instead from nonwage income with wage rate unchanged, it would have a negative income effect on work hours, as illustrated by point F on the wage line EG. The higher wage rate also has a positive substitution effect, offsetting the income effect, and the combined result is a new equilibrium point at Q. As shown, the overall effect of an increase in wage rate is an increase in work hours. A falling or backward-bending labor supply curve would also be possible, of course, given a relatively weaker substitution effect. Again, the illustration of the separate effects is only approximate, because the changes are finite. The simple decomposition of the wage response into income and substitution effects given in equation (8) holds exactly only for infinitesimal changes.

B. Estimation Approximations, Simulation Approximations, and Implicit Assumptions About the Functional Form of Utility

We turn now to a consideration of the properties of underlying utility functions that are implied by a variety of functional approximations employed in empirical studies of labor supply. Virtually any functional form may have reasonable implications if it is applied to a sufficiently restricted range of variables. In practice, however, functional approximations for parameter estimation are employed over nonnegligible ranges of independent variables. Furthermore, many of the more important policy inferences to be drawn from labor supply estimates, such as the simulation of responses to a negative income tax, involve extrapolation of responses outside the range of data used for estimation. It is thus central to proper evaluation of empirical studies that employ approximate supply functions that we consider the utility implications of those approximations over the full relevant range of independent variables.

In the following discussion we shall focus on a variety of single-worker supply functions and consider two broad aspects of their utility implications: theoretical acceptability and general plausibility. Under the first criterion we will seek to define the regions in which the substitution effect is nonnegative and in which leisure and market goods have positive marginal utilities and normal income effects. If these conditions fail within the range of variables relevant to estimation or simulation, we seriously question the theoretical acceptability

of the functional approximation. Evaluation of the plausibility of the implied utility function is necessarily more subjective but helps to place useful perspective on the interpretation of estimated parameters. Such evaluation is also a prerequisite to the consistent use of estimated labor supply parameters in the simulation responses to altered income and earnings opportunities.

Consistent use of estimated labor supply parameters for simulation requires that the same implicit assumptions about functional form of utility be employed for both the estimation and the simulation stages. Such consistency is assured if the estimated supply function is used directly as the simulation equation. That is, if an individual's expected equilibrium labor supply position before a policy change is given by $\hat{H}(w_0, I_0)$, and if the new policy results in a new net budget constraint defined by w' and I' , then the estimated supply function implies a simulated labor supply position at $\hat{H}(w', I')$. However, this procedure has been widely ignored, despite, or perhaps because of, its simplicity.¹⁰ In the majority of studies based on the general model, including one by the present author (1974), the supply function has been used to obtain point estimates of the substitution effect or its elasticity at the population mean. That parameter, assumed to be constant, has then been used for the simulation of responses. In addition to unnecessary complexity,¹¹ this procedure often leads to an inconsistency in the system because the assumed constancy of the substitution effect, or its elasticity, typically implies an underlying utility function different from that implied by the estimated supply function.

The Functional Form of the Substitution Effect

Any labor supply model expressed in explicit analytical form lends itself to a simple derivation of the functional form of the implied substitution effect. Given that the functional form of labor supply (12a) is known, the functional forms of the observable wage and income responses (12b, 12c) follow by simple differentiation.

$$\begin{aligned} (12) \quad a) \quad & H = H(w, I) ; \\ b) \quad & \frac{\partial H}{\partial w} = \frac{\partial H(w, I)}{\partial w} ; \\ c) \quad & \frac{\partial H}{\partial I} = \frac{\partial H(w, I)}{\partial I} . \end{aligned}$$

The substitution effect depends on the observable wage and income responses and on the "point of compensation." The latter value is simply the equilibrium level of labor supply as given by the supply function itself. The functional form of the substitution effect thus follows from the substitution of the functions (12a)-(12c) into equation (9).

$$(13) \quad S(w, I) = \frac{\partial H(w, I)}{\partial w} - \frac{\partial H(w, I)}{\partial I} \cdot H(w, I) .$$

Our treatment of the point of compensation and the substitution effects as functions rather than point values contrasts with most previous applications of the general labor supply model. The functional treatment is necessary to establish the broader properties of the preference structures implied by various assumed estimation forms.

Linear Additive Labor Supply Models

The linear additive supply function is the simplest explicit form, and it has been widely employed in empirical studies.¹²

For this model the supply function is given by (14).

$$(14) \quad H(w, I) = H_0 + Aw + BI.$$

The substitution effect from (13) is given in equation (15).

$$(15) \quad \begin{aligned} S(w, I) &= A - B H(w, I) \\ &= A - BH_0 - BAw - B^2 I. \end{aligned}$$

To establish the limits on the range of wage rates and income for which the substitution effect has the acceptable sign, we simply set $S(w, I)$ equal to zero in (15) and solve for the combinations of values w^* , I^* that define the limits.

$$(16) \quad a) \quad I^* = \frac{1}{B^2} (A - BH_0 - BAw^*);$$

or

$$b) \quad w^* = \frac{1}{BA} (A - BH_0 - B^2 I^*).$$

The equations (16a) and (16b) are simple reciprocals. They describe the combinations of exogenous variables that yield labor supply equilibria at points where the substitution effect is zero. The implications for the underlying preference structure become somewhat clearer when we solve for the labor supply and market consumption points at which this occurs. For labor supply, we substitute into equation (14) from (16) and solve in (17).

$$\begin{aligned}
 (17) \quad H^*(w^*, I^*) &= H_0 + Aw^* + BI^* \\
 &= H_0 + Aw^* + \frac{A}{B} - H_0 - Aw^* \\
 &= \frac{A}{B} .
 \end{aligned}$$

For the critical values of market consumption, we note that the equilibrium value of market goods may be taken from equation (5).

$$(18) \quad M(w^*, I^*) = w^* \cdot H(w^*, I^*) + I^* .$$

Further substitution from (17) and (16) then yields the solution.

$$\begin{aligned}
 (19) \quad M^*(w^*, I^*) &= w^* \frac{A}{B} + I^* \\
 &= w^* \frac{A}{B} + \frac{A}{B^2} - \frac{H_0}{B} w^* \frac{A}{B} \\
 &= \frac{1}{B} \left(\frac{A}{B} - H_0 \right) .
 \end{aligned}$$

Both I^* and M^* are constants. That is, all pairs of values of w^* and I^* that imply a zero substitution effect also imply labor supply and market consumption equilibria at a single point. As will be elaborated below, this implies that the expansion paths through all initial equilibrium points converge at the given point (H^*, M^*) . For want of a better term, we might say that the linear additive supply function implies a concentric homothetic utility function. All indifference curves are identical except for a scale factor that becomes successively smaller the closer the curves are to the bliss point H^*, M^* . We use the term "bliss point" because the indifference curves assume reverse curvature in the region above and to the right of H^*, M^* , implying that that point represents the maximum attainable level of utility.

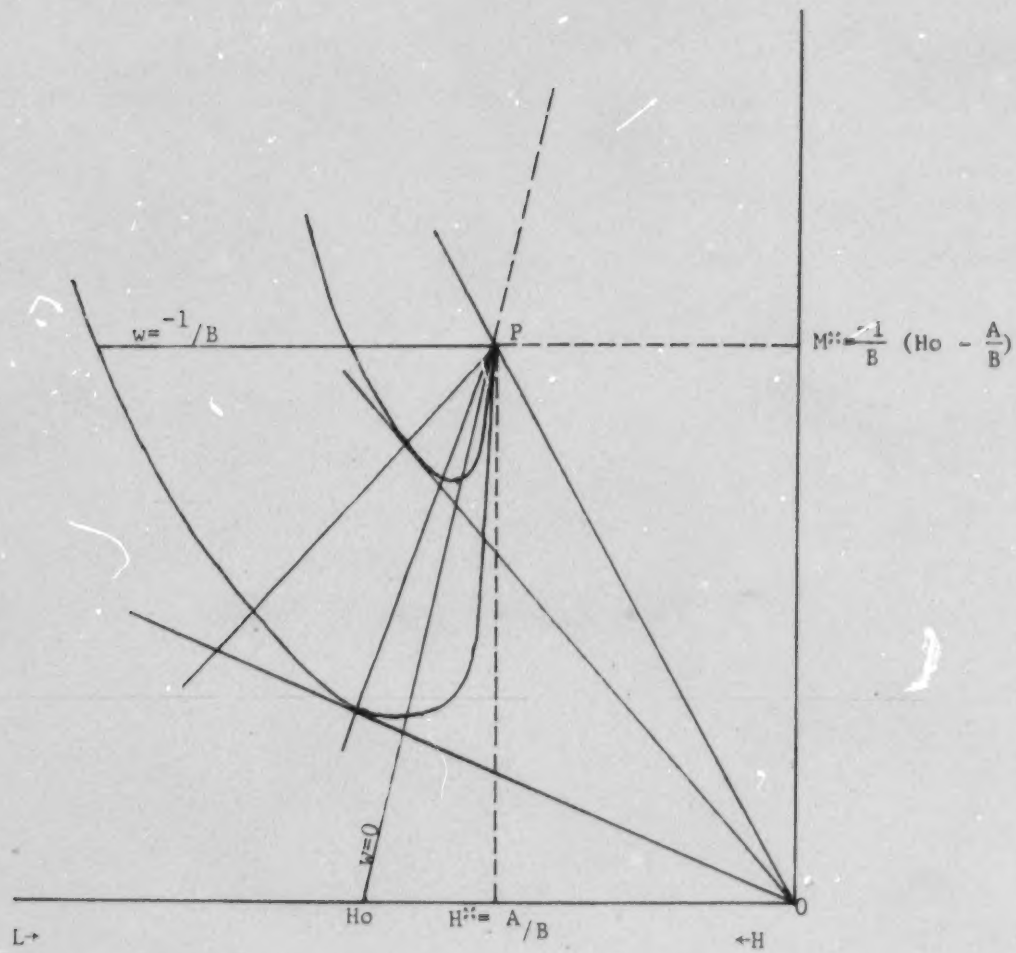
Assuming that B , the income effect for work hours, has the acceptable negative sign as required if leisure is a normal good, H^* will be opposite in sign from the wage coefficient A . Thus for rising curves (A positive) the implied substitution effect takes on unacceptable values only in the irrelevant region of negative work hours. For negatively sloping supply curves, however, H^* is positive, and it becomes important to determine whether the substitution effect goes to zero within the relevant range of other variables. If the point of convergence is well removed from the range of variables over which the linear approximation to the supply function is employed, the implied gradual decline in the substitution effect at higher utility levels may not be a serious problem. If the convergence point is close to or within the important range of variables, however, we must seriously question the usefulness of the approximation.

The convergence properties of the preference map for a linear additive supply function are easily illustrated in the expansion path diagram shown in Figure 2. The diagram also illustrates the boundaries of the region beyond which either leisure or market consumption fails to be a normal good.

An expansion path is the locus of equilibrium values of work hours and market consumption as the budget level is changed while holding the marginal rate of substitution constant. The points on such a locus may be obtained by evaluating the labor supply function and the market demand function at a fixed wage rate and different levels of nonwage income. The slope of an expansion path is determined by the ratio of the income effects for work hours and for market goods.

Figure 2. Expansion Paths for a Linear Additive Labor Supply Function

$$(H = H_0 + Aw + BI ; A, B < 0)$$



$$(20) \quad a) \quad \frac{\partial H}{\partial M} = \frac{\frac{\partial H}{\partial I}}{\frac{\partial M}{\partial I}} = \frac{\frac{\partial H}{\partial I}}{w \frac{\partial H}{\partial I} + 1};$$

$$b) \quad \frac{\partial M}{\partial H} = \frac{\frac{\partial H}{\partial I}}{\frac{\partial H}{\partial M}} = \frac{w \frac{\partial H}{\partial I} + 1}{\frac{\partial H}{\partial I}}.$$

In the linear additive supply model, the income effect on work hours is given by the constant, B . The expressions in (20) thus simplify to those in (21).

$$(21) \quad a) \quad \frac{\partial H}{\partial M} = \frac{B}{Bw+1};$$

$$b) \quad \frac{\partial M}{\partial H} = \frac{Bw+1}{B}.$$

The two alternative expressions in (20a) and (20b) and in (21a) and (21b) are simply reciprocals. They are useful because one of the two may be undefined. Such a case occurs in the present model at a sufficiently high wage level, $w = -1/B$. At that wage rate, an increase in nonwage income is spent entirely on leisure and the income effect on market consumption is zero. The expression in (21a) is undefined, while that in (21b) is zero. The expansion path for this wage rate is represented by the horizontal line to the left of point P in Figure 2. This path marks the boundary of the region in which market consumption is a normal good and thus places a limit on the range of wage rates for which the linear additive model is an acceptable approximation.

The fact that the expansion-path slope, $\frac{\partial H}{\partial M}$, varies with wage rate in this model while the income effect, B , is a constant,

underscores a distinction that has not always been clear in the labor supply literature. A confusion of the two parameters appears to be the source of the inconsistency in the Ashenfelter-Heckman model discussed below. Cain and Watts (1973) also appear to refer to the two parameters interchangeably.¹³

The lower boundary of the region for which the linear additive model is an acceptable approximation is given by the zero-wage expansion path. This path extends from H_c to the convergence point P in Figure 2. Values of work hours to the right of this locus are achieved only at negative wage rates, with the implication that the marginal utility of leisure is negative in that region.

It is of interest to consider the boundary values implied by actual estimates of labor supply parameters in this model. Linear, additive labor supply functions estimated in three representative studies are presented in Table 1. We have chosen only examples in which the point estimate of the substitution effect at the mean values of work hours, wages, and income has the acceptable sign. At this juncture we are concerned only with internal consistency of the parameters and do not consider possible biases or other estimation problems.

The critical values of work hours and total family income (H^* and M^*) at which the implied substitution effect goes to zero are shown in the first two columns of Table 2. The other entries in the table help to place these critical values in perspective. The third column, $W^*(0)$, gives the wage rate at which this equilibrium is reached if there is no nonwage income. The fourth column,

Table 1. Representative Examples of Estimated Linear Additive Labor Supply Function

<u>Greenberg-Kosters</u> ^c			
Working male family heads			
1. Simple linear model	$H = 2400^a$	$-108.2w$	$-.0675I$
Model with kinked wage curve			
2. Wage rate under \$2.00/hr.	$H = 2340^a$	$-74w$	$-.066I$
3. Wage rate over \$2.00/hr.	$H = 2400^a$	$-103w$	$-.066I$
<u>Hill</u> ^d			
4. Poor white males	$H = 2300^a$	$-204w$	$-.40I$
5. Poor black males	$H = 2200^a$	$-154w$	$-.32I$
6. Nonpoor white males ^b	$H = 2700^a$	$-172w$	$-.215I$
<u>Dickinson</u> ^e			
Working male family heads			
7. 5-year average work hours	$H = 2560$	$-53w$	$-.061I$
Working wives			
8. 5-year average work hours linearized step function	$H = 800$	$+400w$	$-.03I$

^aThe intercept term is approximate because numerous control variables whose means were unreported were included in the estimation equation. We have regarded the model as a simple two-predictor model and have chosen the intercept to yield 2000 hours at the mean wage rate and zero income. The choice is not critical to our interpretation of results, since the degree of approximation is small in proportion to total annual work hours.

^bThis model has been simplified to eliminate a weak wage-income interaction, as discussed in the text.

^cGreenberg and Kosters (1970), also in Cain and Watts (1973).

^dHill (1973).

^eDickinson (1974).

Table 2. Variable Values Implying Zero Substitution
Effects in Representative Labor Supply Functions

	H*	M*	w*(C)	I*(\$2/hr)
<u>Greenberg-Kosters^a</u>				
1. Simple linear model	1600	\$11,852	\$ 7.41	\$ 8,652
2. Kinked wage effect under \$2	1121	\$18,466	\$16.47	\$16,224
3. Kinked wage effect over \$2	1560	\$12,718	\$ 8.15	\$ 9,598
<u>Hill^b</u>				
4. Poor white males	510	\$ 4,475	\$ 8.75	\$ 3,455
5. Poor black males	475	\$ 5,400	\$11.35	\$ 4,450
6. Nonpoor white males	800	\$ 8,838	\$11.04	\$ 7,238
<u>Dickinson^c</u>				
Males, 5-year average	868	\$27,724	\$3,194	\$25,988
Wives, 5-year average	-13,333	471,110	--	--

^aGreenberg and Kosters (1970).

^bHill (1973).

^cDickinson (1974).

$I^*(\$2/\text{hr.})$, gives the level of nonwage income that will again yield the critical equilibrium at a wage of \$2.00 per hour.

The simple linear Greenberg-Kosters supply function was estimated for working married men with family incomes under \$15,000 per year. The critical value of market income, \$11,852, thus falls well within the range to which the approximate supply function is intended to apply. In the Greenberg-Kosters model estimated with a kinked wage slope, the segment below \$2/hr. has a weaker negative slope. That slope, together with the single estimate of the income slope, implies a convergence point slightly outside the sample range. The segment above \$2/hr. is similar to the simple linear model and implies a similar convergence point.

The supply functions estimated by Hill for poor males, black and white, have much stronger income effects and more steeply backward-bending wage slopes. The values of total family income at which the implied expansion paths converge are low, but, given the sample restriction to families below the poverty line in 1967, they are outside the sample range for all but large families. The model estimated by Hill for nonpoor whites includes interaction terms between wage rates and income, but these coefficients are small relative to the wage and income effects. The effect of the interaction is evaluated at the mean wage rate and is incorporated into the income coefficient shown in Table 1, but otherwise is ignored. Thus simplified, the supply function implies indifference curves that are concentric about a market consumption point under \$9,000, which cannot be considered reasonable for nonpoor white males

in 1967. Taking full account of the small estimated interaction effect would raise the value slightly and eliminate the single point of convergence but would not materially change the implications.

The linear supply function estimated by this author for average work hours for male family heads who were in the labor force continuously from 1967-1971 has an income effect comparable to that in the Greenberg-Kosters study and a more weakly negative wage slope. The implied convergence point is thus at a level of total income above \$25,000. There is no income cut-off in the sample selection, however, so this point falls within the top one or two percentiles of the distribution, and the implied variation in the substitution effect over the sample range is quite large.

The rising supply function estimated for wives in the same study is included for contrast. The implied indifference curves are also concentric, but the point of convergence is so far outside the sample range that the within-range variation in the shape of the utility contours is slight.

The expansion path boundary at which market consumption becomes inferior may be quite restrictive in models with strong estimated income effects. Hill's estimates for poor white males, for instance, imply an upper bound of $w = \$2.50$ on this criterion. In the Greenberg-Kosters model, however, this upper bound is roughly \$15.00. This value is less restrictive than the convergence wage, $w^*(0)$.

Negative wage rates are not in the relevant range, either for estimation or simulation, so the lower boundary does not imply unacceptability from a strictly theoretical point of view. However, the

fact that the zero-wage equilibria implied by many estimates occur at levels of work hours close to standard full-time hours is subject to serious question on grounds of general implausibility. At zero wage and \$6000 income, for instance, the Dickinson estimates imply 2194 hours worked per year and the Greenberg-Kosters estimates imply 1950 hours worked per year. While it is possible that the first 40 or so hours per week spent at work are more intrinsically rewarding than any alternative leisure activities, it is generally presumed that a substantially greater proportion of time would be spent in leisure if there were no cost in forgone earnings.

Additive Curvilinear Supply Functions

The boundaries of the acceptable regions implied by simple curvilinear additive supply functions may be derived by straightforward extensions of the above results. We present the solution for a function with a quadratic wage effect as an illustration.

$$(22) \quad H(w, I) = H_0 + A_1 w + A_2 w^2 + BI.$$

$$(23) \quad S(w, I) = A_1 + 2A_2 w - BH(w, I)$$

$$= A_1 + 2A_2 w - B(H_0 + A_1 w + A_2 w^2 + BI).$$

Setting $S(w, I)$ equal to zero and solving for the critical points (w^*, I^*) , we obtain

$$(24) \quad I^* = \frac{1}{B^2} [-BA_2 w^{*2} + (2A_2 - BA_1) w^* + A_1 - BH_0].$$

The quadratic supply function no longer implies a single convergence point in the leisure-goods plane but rather a locus of

such points. The locus will be in the region of negative work hours for rising portions of the quadratic wage function and in the region of positive work hours for declining portions.

Estimated parameters for two examples of this type of quadratic supply function are presented in Table 3a. Two points on the convergence locus are presented for illustration; that obtained if the wage is increased and income is held constant at zero, and that obtained if the wage rate is held constant at \$2.00 per hour and income is increased. In both cases shown, the increasing backward bend of the supply function implies that the substitution effects go to zero well within the sample range. These models imply linear expansion paths as a consequence of the linear income effect. The upper boundary at which market goods become inferior occurs at $w = \$6.67$ in Rosen and Welch's model and at half that value in Hill's estimates. The zero-wage expansion paths imply somewhat lower zero-wage work hours than those in the simple linear models, but those values are still more than half of standard full-time work hours over much of the range of income.

The Kalachek and Raines estimates presented in Table 3b provide an example of a rising supply curve for which the substitution effect does not go to zero within the relevant range. The comparatively weak estimated income effect implies inferiority for market goods only at wage rates in excess of \$20.00 per hour. The income effect is quadratic and becomes weaker at higher levels of nonwage income, so it implies inferiority for leisure at incomes above \$2500. The zero-wage locus is also troublesome, in that it implies zero-wage equilibria always above 35 hours per week.

Table 3. Selected Empirical Quadratic Labor Supply Functions with Examples of Critical Points

a. Quadratic wage effects

Rosen and Welch: urban males^b

$$H = 2060^a + 255w - 79w^2 - 15 I$$

Convergence points (zero substitution effect)	I*	w*	H*	M*
	0	\$3.49	1976	\$6,938
	\$13,900	\$2.00	169	\$14,489

Zero-wage locus $H = 2060 - .15 I$

Inferiority of market goods $w > \$6.67$

Hill: poor black males^c

$$H = 1740^a + 495w - 217w^2 - .31 I$$

Convergence points (zero substitution effect)	I*	w*	H*	M*
	0	\$2.36	1707	\$4,030
	\$6,970	\$2.00	-289	\$6,392

Zero-wage locus $H = 1740 - .31 I$

Inferiority of market goods $w > \$3.23$

b. Quadratic income effect

Kalachek and Raines: lower-income males aged 24-61^d

$$\text{Hours per week} = 37.1^e + 3.12w - .89 \times 10^{-3} I + 1.77 \times 10^{-7} I^2$$

$$\text{Zero-wage locus } H/\text{wk} = 37.1 - .89 \times 10^{-3} I + 1.77 \times 10^{-7} I^2$$

$$\text{Minimum hrs/wk} = 36 \text{ hrs/wk at } I = 2500$$

^a Intercept approximated as in Table 1.

^b Rosen and Welch (1971).

^c Hill (1973).

^d Kalachek and Raines (1970).

^e Intercept chosen to yield $H=44.2$ at $w=\$2.40$ and $I=\$458$, the mean values.

Estimation-Simulation Inconsistencies

The implausibly high zero-wage equilibrium work hours implied by many estimated supply functions may be one of the reasons researchers have introduced a second set of assumptions for simulation of labor supply responses. The most common assumptions for the simulation stage are that the substitution effect is a constant¹⁴ or that the elasticity of the substitution effect is constant.¹⁵ These assumptions differ, sometimes dramatically, from the form of the substitution effect implicit in the estimating function. The sharpest inconsistency occurs in the conversion from a simple linear estimating form to a constant-elasticity simulation form. In the simple linear form, the implied substitution effect is a linearly increasing function of the level of work hours, as may be seen from the first expression of (15).¹⁶ A constant substitution elasticity less than unity implies precisely the opposite variation.¹⁷ The assumption of a constant substitution effect is intermediate between the two. The negative quadratic functions estimated by Rosen and Welch and by Hill yield variation in the substitution effect that is consistent in direction with the constant-elasticity assumption, but substantially weaker in magnitude at low marginal wage rates.

Logarithmic Estimation Form and the Substitution Elasticity

When considered in the light of predicted responses to very low marginal wage rates, a substitution effect having a constant elasticity seems much more plausible than the constant substitution effect or

that implied by a linear supply model. Small changes in the wage rate as it approached zero would be increasingly large in proportional terms and would imply large reductions in equilibrium work hours. With this desirable feature in mind, it is tempting to estimate a labor supply model in double-log form.

$$(25) \quad \log H = K + \alpha \log w + \beta \log I.$$

This functional form has the familiar property that responses of work hours to changes in wage rates or income are constant in elasticity, with the elasticities given by the estimated values of α and β , respectively.

$$(26) \quad a) \quad \frac{\partial H}{\partial w} \cdot \frac{w}{H} = \alpha ;$$

$$b) \quad \frac{\partial H}{\partial I} \cdot \frac{I}{H} = \beta .$$

Unfortunately, this functional form, which provides so simply for constant elasticities with respect to observable changes, does not produce a substitution effect that is constant in elasticity. On the contrary, when the wage effect is negative the substitution effect goes to zero faster with decreasing work hours in the log-log case than it does in the simple linear model. The expression for the substitution effect follows from (13) and (26a) and (26b).

$$(27) \quad S = \frac{\partial H}{\partial w} - H \frac{\partial H}{\partial I} \\ = \alpha \frac{H}{w} - H \beta \frac{H}{I} \geq 0 .$$

The elasticity of the substitution effect,¹⁸ which we denote η_s , is given by a slightly simpler expression.

$$(28) \quad \eta_s = \left. \frac{\partial H}{\partial w} \right|_u \cdot \frac{w}{H} = S \frac{w}{H} = \left(\alpha - \beta \frac{wH}{I} \right) \geq 0 .$$

The variation in the elasticity of the substitution effect is most easily seen from the last expression of (28). The second term, $-\beta \frac{wH}{I}$, is positive, since β is negative, but declines rapidly in the face of a compensated reduction in wage rates. Under such a reduction, w and H decrease and I increases, changes that all contribute to the decrease in the positive term. A theoretically acceptable positive substitution effect is maintained only so long as the ratio of earnings (wH) to nonwage income exceeds the ratio of the wage elasticity to the income elasticity.

$$(29) \quad \frac{wH}{I} > \frac{\alpha}{\beta} \iff \eta_s \geq 0 .$$

This condition is always met if the wage elasticity is positive, since β is negative making the ratio α/β negative. For the backward-bending supply curve typically estimated for male heads of families, however, the critical values are restricting. The most favorable estimates of Kusters (1966), using a double logarithmic form, yield values of $\alpha = -.062$ and $\beta = -.023$. Given these estimates, nonwage incomes of greater than 37 percent of earnings imply a negative substitution effect.

The Ashenfelter-Heckman Model

Ashenfelter and Heckman (1973) suggest a model intended to maintain consistency between the functional forms employed in estimation and simulation. Unfortunately, their two basic assumptions, a constant substitution effect and a constant income effect, are

themselves mutually inconsistent. The inconsistency is easily demonstrated if we assume only a constant income effect, B , and obtain an expression for the substitution effect at different income levels. At initial wage and income levels (w_0, I_0) , the substitution effect is given by (30).

$$(30) \quad S(w_0, I_0) = \frac{\partial H(w_0, I_0)}{\partial w} - B H(w_0, I_0) .$$

If income is changed from the initial level by an amount ΔI , the wage and income derivatives on the right-hand side of (30) are unchanged,¹⁹ but equilibrium work hours change by an amount $B\Delta I$. The substitution effect at the new income level is thus given by (31).

$$\begin{aligned} (31) \quad S(w_0, I_0 + \Delta I) &= \frac{\partial H(w_0, I_0 + \Delta I)}{\partial w} - B H(w_0, I_0 + \Delta I) \\ &= \frac{\partial H(w_0, I_0)}{\partial w} - B H(w_0, I_0) - B\Delta H(\Delta I) \\ &= S(w_0, I_0) - B^2 \Delta I . \end{aligned}$$

A constant income effect and a constant substitution effect are thus consistent only in the degenerate case of $B = 0$.

The inconsistent theoretical assumptions of Ashenfelter and Heckman's model are not incorporated into their actual empirical work. The estimation model they employ results in constancy, not of the income effect, but of the slope of the expansion path, $\partial H / \partial M$. When the latter derivative is taken to be a constant parameter, which we denote D , the form of the income effect is given by inversion of (20a).

$$\begin{aligned}
 (32) \quad \frac{\partial H}{\partial I} &= \frac{\frac{\partial H}{\partial M}}{1 - w \frac{\partial H}{\partial M}} \\
 &= \frac{D}{1 - wD}
 \end{aligned}$$

The Ashenfelter-Heckman estimates of D are of the order of $-.066$, which implies income effects of $-.06$ at $w = \$1.50/\text{hr.}$ and $.044$ at $w = \$7.50/\text{hr.}$ The Ashenfelter-Heckman method also results in a substitution effect that is a decreasing function of the marginal wage rate, rather than a constant as assumed in the theoretical model. Both properties are treated more fully in the general discussion of parallel utility functions in section III.

C. A General Empirical Model

We turn finally to the flexible functional form employed by Cohen, Rea, and Lerman (1970) and Rea (1971). They estimate labor supply response as a step function with multiple interactions between wage rates and levels of nonwage income. However, some of the generality gained by their estimating procedures is assumed away when they turn to the standard assumptions about the substitution effect for the simulation of labor supply responses to policy changes. Rea focuses more closely on the simulation question, and he does maintain approximate consistency between the two stages. In particular, he revives formulae derived by Hicks (1946) to adapt the infinitesimal relationship of equation (13) to finite changes in income and wage rates. Rea estimates different values of the substitution effect for each of several intervals of wage rate and income but assumes that the substitution effect is constant

within each interval. The assumed stepwise constancy of the substitution effect results in loss of generality because Rea's estimates suggest that the substitution effect is changing rapidly within some of the intervals.

Rea's tabulation of his estimation results makes the illustration of this point particularly convenient. Again, we emphasize that this discussion abstracts from any problems of estimation and is concerned only with the implications of estimates. His Table V-3 provides estimates of equilibrium annual work hours for husbands aged 25-62 whose wives are not working. The estimated work hours for individuals with a standard set of characteristics are given as a step function of wage rates and income with a full set of interactions. The wage function has four steps for the ranges 0 - \$0.99/hr., \$1.00 - \$2.49/hr., \$2.50 - \$4.99/hr., and \$5.00/hr. and above. Within each wage interval, a step function for the income effect is estimated over the following four ranges of nonwage income: 0 - \$499, \$500 - \$1499, \$1500 - \$2999, and \$3000 and above. Rea's procedure is to interpret each difference between adjacent steps of wage rates and income as a segment of a linear supply curve and to calculate a single constant substitution effect, which is presumed to be applicable to that range.

An alternative approach, which, in the opinion of this author, better exploits the generality of the estimating form, entails the generation of an approximate map of the utility surface. As shown in Figure 3, the points representing joint equilibrium values of work hours and market consumption may be plotted for the several income steps within a given wage range and interpreted as an

approximate expansion path for that wage range. This has been done in Figure 3 for all but the highest wage range, in which the estimated income effect has the wrong sign. The mean value of the lowest wage range has also been set at \$.75/hr. rather than at Rea's \$.49/hr., on the grounds that very few prime-age males have wage rates toward the bottom of that range. If we impose the condition that expansion paths for different wage rates may not intersect within the range of sample values, and if we approximate the paths by straight lines,²⁰ Rea's estimated values prove to be remarkably consistent with a homothetic utility function with an origin at \$2000 of total income and T-2875 hours of leisure per year. The approximate expansion paths are shown in enlarged scale in Figure 4, along with indifference curves approximated by wage-line segments. The elasticity of the substitution effect clearly increases dramatically as one moves down and to the right along an indifference curve. The change in hours resulting from compensated wage changes from \$3.75 to \$1.75 per hour is scarcely more than one-tenth the size of the change in hours when the marginal wage drops from \$1.75 to \$.75 per hour. It is reasonable to suppose that this radical increase in the substitution effect is distributed with some uniformity over this interval. Rea, however, estimates a substitution effect on the basis of the stronger income effect at the low-wage end of each interval and uses that estimate to simulate all labor supply responses in that interval. For workers near the upper end of the lowest wage interval, this results in a several-fold overestimate of the substitution effect at the equilibrium point. The cumulative response to a substantial compensated wage change will also be overestimated, but not so seriously.

Figure 3. Approximate Expansion Paths Based on Rea's Estimates

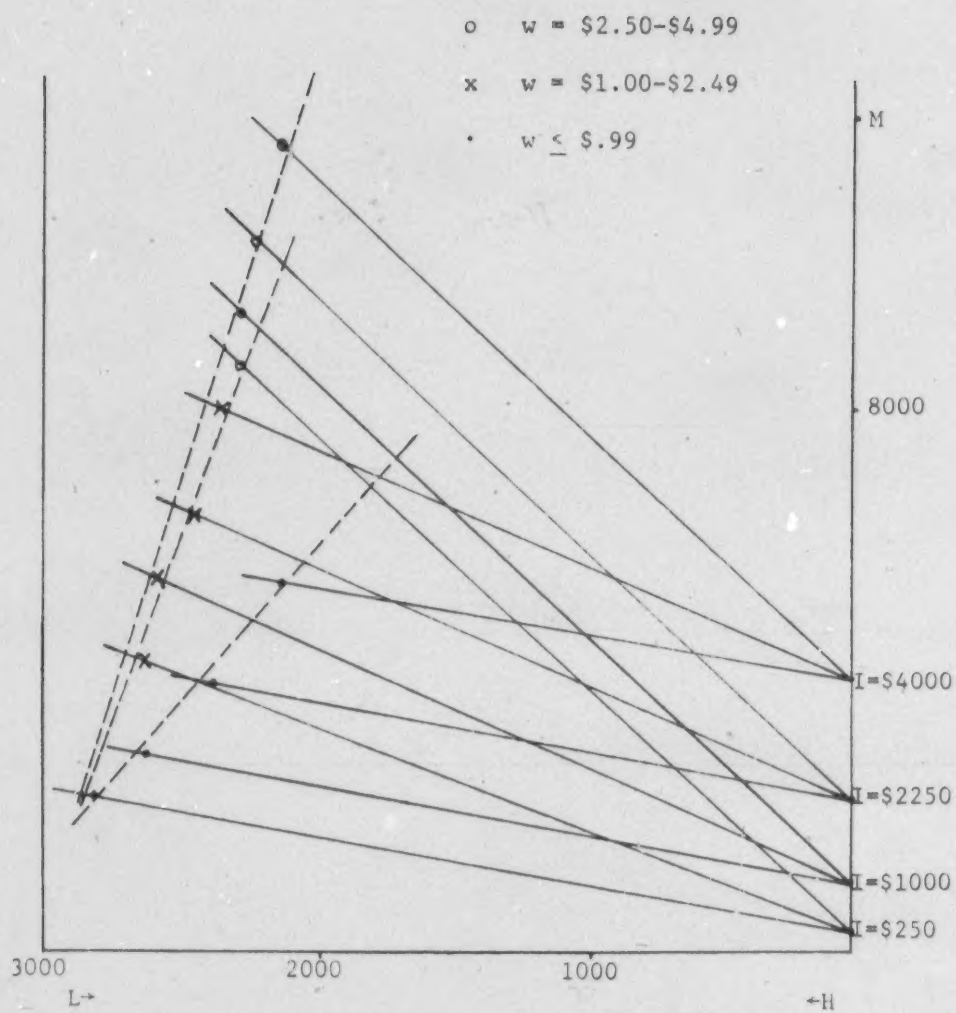
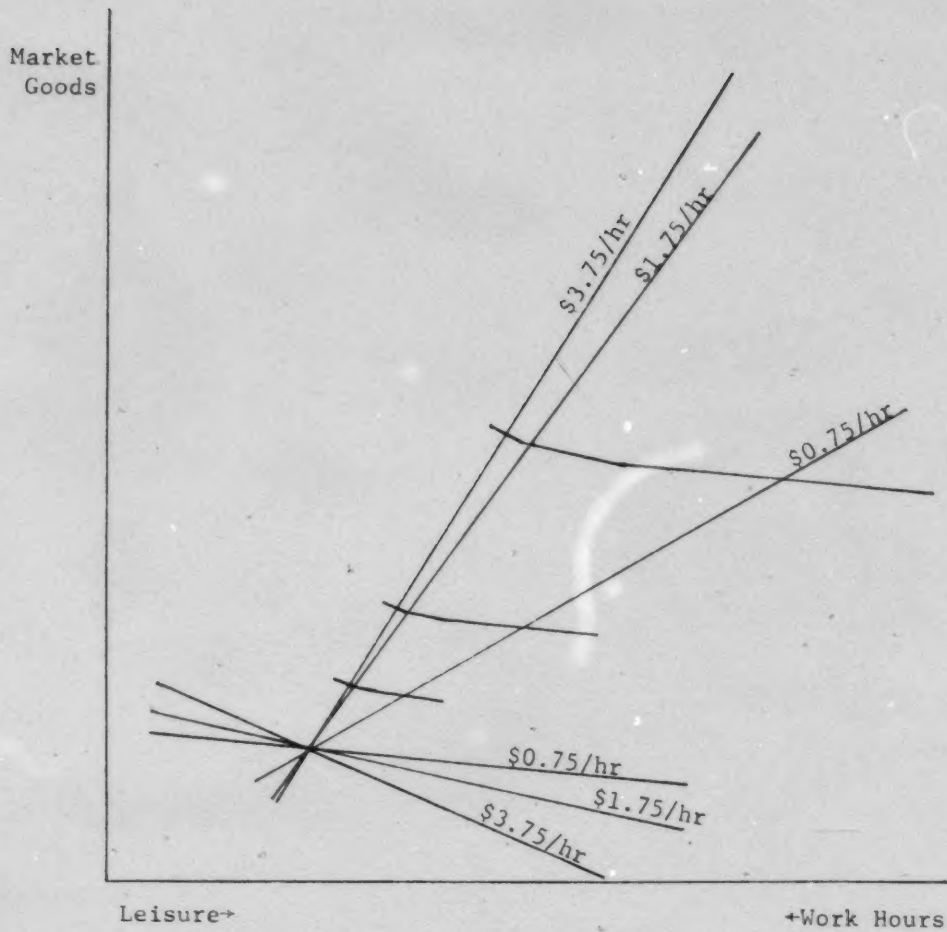


Figure 4. Approximate Indifference Map Based on Rea's Estimates



The approximate utility map that follows directly from the pattern of estimated expansion paths is one of the primary advantages of this interpretation of estimated parameters from the general interactive functional form. Under reasonable continuity assumptions, the numerous point estimates of partial derivatives fall into place as integrated properties of a continuous function defined over a substantial range of variables. Furthermore, the basic properties of both the supply function and the implicit utility function are directly evident by inspection.

The general pattern of expansion paths estimated under the Cohen, Rea, and Lerman model may be used for tests of the appropriateness of alternative analytical labor supply models. The preceding exposition of the properties of implicit preference structures provides part of the theoretical background for those tests. To complete the picture, we now turn to a consideration of the preference structures for a variety of explicit utility models of labor supply.

III. Labor Supply Models Based on Explicit Utility Functions

A. Introduction

Growing numbers of labor supply studies have been based on models derived from explicit assumptions about the form of the utility function for leisure and market consumption. This approach has both strengths and weaknesses as compared with the more common approach based on general demand theory discussed in the previous

section. On the positive side, functional consistency between estimation and simulation is easily and naturally maintained. Furthermore, if the estimated parameters are theoretically acceptable anywhere within the relevant range of variables, the implied preference structure is usually acceptable and consistent over a large range. However, the virtues of functional consistency and regularity are often gained at the expense of questionably arbitrary restrictions, which are implicit in the assumed functional form of utility. In some cases these restrictions imply implausible labor supply behavior, which becomes still more extreme if the estimated functions are extrapolated beyond the range of data in the estimation sample.

In this section we review several of the common utility functions that have been employed in labor supply studies. Our approach will have two general aspects. For general perspective we will consider the plausibility of behavior implied by each function, particularly in the range of budget variables relevant to negative-income-tax simulations. For specific relevance to our proposed tests of alternative preference structures we will characterize each utility function in terms of its implied pattern of expansion paths. These results will provide a framework for interpretation of the patterns of expansion paths generated by our subsequent estimates using the general interactive labor supply model.

The discussion is limited to single-worker utility functions. We also restrict attention to those functions having linear expansion paths in the plane of leisure and market consumption. This restriction

has the substantial benefit of simplicity, particularly when the model is complicated by marginal tax rates and overtime premiums.²¹ It is also made in the belief that there is not sufficient variance in available nonwage-income data to allow estimates of income curvature in addition to wage-income interactions. The restriction has the effect of excluding only one common class of utility functions that has received recent attention in the labor supply literature--namely, the "trans log" utility function.²² The simpler functions more commonly applied in labor supply studies all have the linearity property. These include the Cobb-Douglas, the Stone-Geary, and the C.E.S. functions as well as the quadratic utility function and those in a general class that we shall refer to as parallel utility functions.

In considering the labor supply behavior implied by the various utility functions, we will describe the budget constraint by a simple wage rate, constant for all work hours, and a level of nonwage income ($M = WH + I$). For problems with complex budget constraints in which net wage rates and income are affected by progressive marginal tax rates, overtime premiums, and the like, the complex net budget constraints will be replaced by transformed simple budget constraints that are equivalent in the neighborhood of equilibrium positions.

Our procedure of describing labor supply behavior as a function of equivalent simple budget parameters differs from that used in several studies that assume specific utility functions. The alternative procedure involves including tax parameters in the budget specification so that they also appear as arguments in the solution functions for optimal labor supply.²³ While the latter

procedure is convenient for specific questions about particular tax and incentive structures, it is cumbersome for purposes of general inference about preference structures.

B. Specific Utility Functions

Linear Expenditure System Utility Function

In consumer demand studies based on specific utility functions, one of the more familiar models is the Stone-Geary modification of the Cobb-Douglas function. For a single-worker family with utility dependent on market goods and the worker's leisure, the utility function has the form of equation (33).

$$(33) \quad U = K(L-\lambda)^{\alpha} (M-\mu)^{\beta} .$$

The parameters λ and μ are interpreted as minimum subsistence levels of leisure and market goods, respectively. The single-worker form of equation (33) has been used in a study by this author (1970). Leuthold (1968) employed an expanded version for two-person families, and Christensen (1971) introduced intertemporal optimization for single-worker families. Horner (1973) used a Cobb-Douglas function without subsistence parameters in an analysis of labor supply responses in the New Jersey Income-Maintenance Experiment. In the present discussion we will focus on the form of equation (33).

For models with a given functional form for utility, the derivation of the labor supply function is very straightforward. We simply substitute the budget identity, ($M = wH + I$), and the identity relating leisure and work hours, ($L = T - H$), into the utility function, differentiate with respect to H , and solve for the optimal value as a function of w and I . In the present case, our notation is simplified by

defining a new parameter, $\tau = T - \lambda$, the total time available in excess of subsistence-leisure requirements.

After the substitutions the utility function takes the form of equation (34).

$$(34) \quad U = k(\tau - H)^\alpha (wH + I - \mu)^\beta.$$

Differentiating a logarithmic form and solving for optimal work hours yields the supply function in the alternative forms (35c) and (35d).

$$(35) \quad a) \quad \frac{\partial \log U}{\partial H} = \frac{\partial}{\partial H} (\log k + \alpha \log(\tau - H) + \beta \log(wH + I - \mu)) = 0;$$

$$b) \quad \frac{-\alpha}{(\tau - H)} + \frac{w\beta}{(wH + I - \mu)} = 0;$$

$$c) \quad H = \frac{\beta\tau}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} \frac{(I - \mu)}{w} = \beta\tau - \alpha \frac{(I - \mu)}{w}$$

$$d) \quad wH = \frac{\beta\tau}{\alpha + \beta} w - \frac{\alpha}{\alpha + \beta} (I - \mu) = \beta\tau w - \alpha(I - \mu).$$

The forms (35c) and (35d) are equivalent except that the latter is expressed in terms of earnings (wH) rather than work hours. For equation (35d), earnings are seen to be a linear function of both wages and income. This form is a direct analog to the linear expenditure functions employed in Stone's analysis of consumer demand (1954). The additive linearity of the earnings function associated with the homothetic Cobb-Douglas-type utility function is in distinct contrast with the linearity of the labor supply function, which implies a concentric utility function as was shown in section II.

The direct labor supply function (35c) is also simple in form but is clearly not additive. In particular, the magnitude of the income effect is inversely proportional to the wage rate. This implied

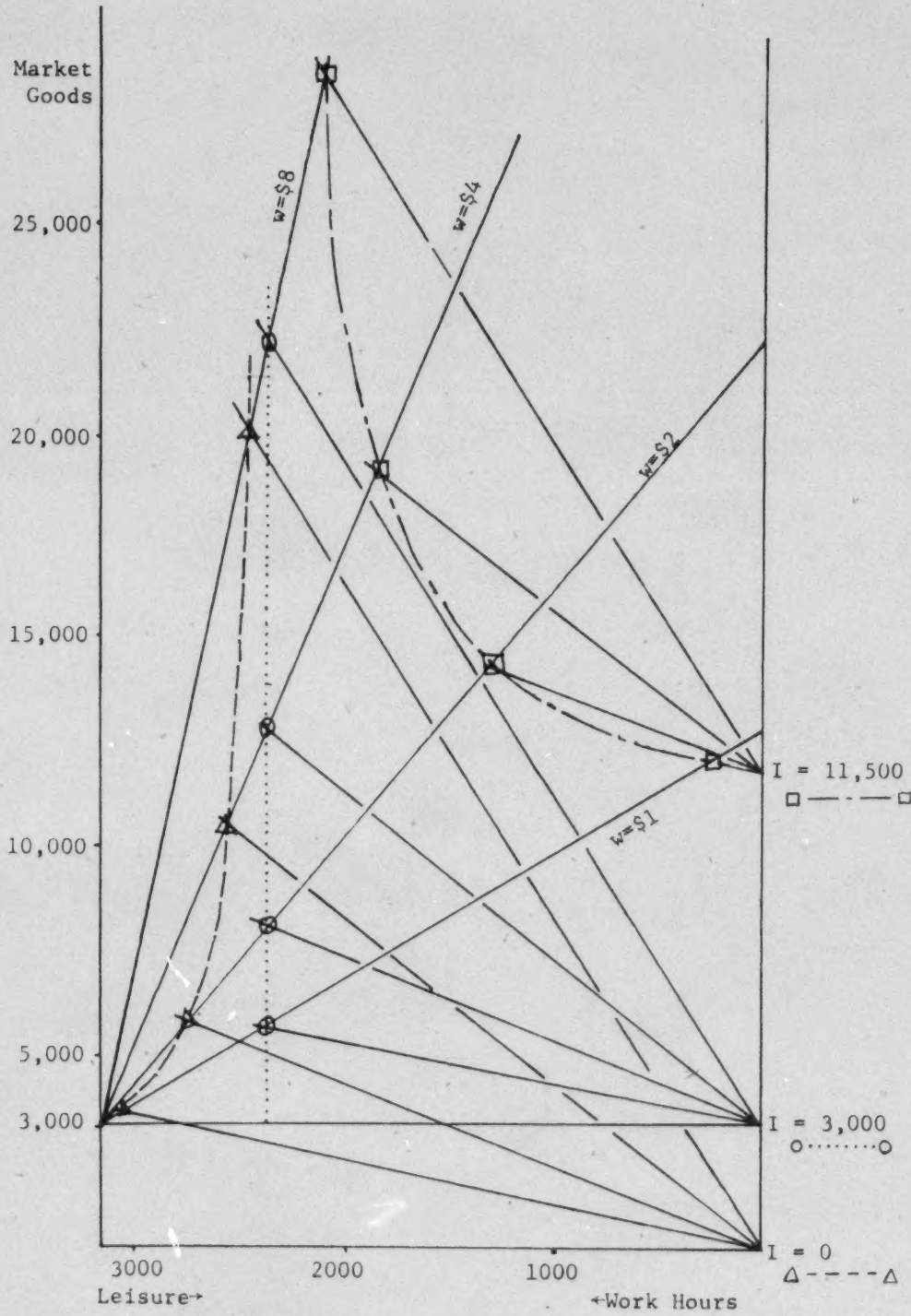
inverse proportionality provides the null hypothesis for a statistical test of the appropriateness of the Stone-Geary functional form. If the estimates from the general interactive model are consistent with the Stone-Geary form, we fail to reject the null hypothesis that the income effects for different wage intervals are inversely proportional to the respective wage rates.

The wage slope may be positive, negative, or zero. If nonwage income, I , is just equal to the subsistence requirement, μ , the labor supply function implied by the Stone-Geary function is completely inelastic, with optimal work hours a constant fraction, β , of the effective time available regardless of wage rate. If the quantity $(I-\mu)$ is positive, optimal work hours increase with wage rate, and, conversely, if $(I-\mu)$ is negative, labor supply decreases with wage rate. In all cases the implied labor supply functions converge to the value $\beta\tau$ in the limit of very high wage rates.

The labor supply responses are illustrated in the expansion-path diagram in Figure 5. Each expansion path represents a locus of optimal labor supply and market consumption points for a given wage rate and varying levels of nonwage income. A particular equilibrium point is then represented by the intersection of a budget line with the corresponding expansion path. A tangency with an indifference curve is implicit at each such point, but indifference curves are omitted for simplicity of the figure.

The expression for any given expansion path is derived by equating the ratio of the marginal utilities of market goods and leisure to a given wage rate. Recall that the wage rate is, in effect, a price ratio, since it is deflated by the price of market goods. Fo

Figure 5. Expansion Paths and Labor Supply Curves for a Stone-Geary Utility Function



the modified Cobb-Douglas functions, the expansion-path functions are given by (36a) and (36b), with the two expressions being simple reciprocals.²⁵

$$(36) \quad a) \quad H = \tau - \frac{\alpha}{\beta w} (M - \mu) ;$$

or

$$b) \quad M = \mu - \frac{\beta}{\alpha} w(\tau - H) .$$

For the illustration in Figure 5 we have chosen the following parameter values: $\alpha = .25$, $\beta = .75$, $\mu = \$3000$, and $\tau = 3200$ hours per year.²⁶ When nonwage income is equal to \$3000, labor supply is a constant 2400 hours per year at all wage rates. For lower income, annual labor supply falls from 3200 hours at zero wage rate and approaches the 2400-hour asymptote at high wage rates. For higher income levels, labor supply rises from zero at low wage rates toward 2400 hours at higher wage rates. If α were larger relative to β , the asymptotic value of work hours would be a smaller fraction of total effective time (perhaps accomplished by increasing τ) and the steeply rising and falling supply curves would be obtained over a wider range of wage rates for any given magnitude of $(I - \mu)$. Conversely, if β were very large relative to α , the asymptote would be a large fraction of τ and the steeply rising and falling supply curves would occur only at very low wage rates unless $(I - \mu)$ were quite large.

Another feature of this model is that combinations of parameters and economic variables are acceptable only if it is possible to attain levels of leisure and market consumption above the subsistence

requirements. In the illustrated case, a wage rate of \$.75 per hour with no outside income would yield subsistence-level market consumption only if leisure were reduced below subsistence levels. The utility and labor supply functions are thus not defined for such values of budget variables.

C.E.S. Utility Functions with Subsistence Parameters

The Cobb-Douglas type of utility function discussed above is one of a class of functions having constant elasticity of substitution (C.E.S.). The more general C.E.S. function with subsistence parameters has been employed by Wales (1973) in a study of the labor supply of self-employed businessmen. As noted earlier, our treatment differs from his in that we do not include tax parameters in the budget constraint or the derived supply equations.

Deviation of the labor supply functions implied by a C.E.S. utility function is simplified if the utility function is expressed as an "addilog" function with equal exponents, as shown in (37).²⁷

$$(37) \quad a) \quad U(L, M) = \alpha(L - \lambda)^{-\rho} + \beta(M - \mu)^{-\rho} ;$$

or

$$b) \quad U(H, w, I) = \alpha(\tau - H)^{-\rho} + \beta(wH + I - \mu)^{-\rho} .$$

Equation (37b) follows from (37a) by substitution of the budget and labor-leisure identities. The parameter, ρ , is related to the elasticity of substitution, σ , as shown in equation (38).²⁸

$$(38) \quad a) \quad \rho = \frac{1}{\sigma} - 1 ;$$

or

$$b) \quad \sigma = \frac{1}{1+\rho} .$$

The solution for the optimal labor supply function is analogous to the Cobb-Douglas case and is accomplished by maximization of (37b) with respect to H . The solution function is given in equation (39).

$$(39) \quad H = \frac{\tau - (\alpha/\beta)^{-\sigma} w^{-\sigma} (I - \mu)}{1 + (\alpha/\beta)^{\sigma} w^{1-\sigma}} .$$

It is easily confirmed that the C.E.S. supply function reduces to the form of (35c) in the special case of $\sigma = 1$. By contrast with the special Cobb-Douglas case, which is linear in the ratio of income to the wage rate and the reciprocal of wage rate, the general C.E.S. form in (39) requires nonlinear estimation techniques for empirical application. It is thus less attractive as a model unless the greater generality is shown to be necessary.

The functional form of the income effect implied by the C.E.S. supply function again forms the basis of our proposed tests of the appropriateness of this functional form. Differentiation of (39) yields the expression in (40).

$$(40) \quad \frac{\partial H}{\partial I} = \frac{-w^{-\sigma}}{(\beta/\alpha)^{\sigma} + w^{1-\sigma}} .$$

Any hypothesized values of σ and the ratio (β/α) will imply ratios of income derivatives for different wage intervals, which may be tested using the estimates from the interactive model.

The overall pattern of labor supply behavior implied by C.E.S. utility functions is again clarified by illustration in an expansion-path diagram. The expressions for an expansion path, derived by equating the marginal rate of substitution to the given wage rate, are shown in equations (41a) and (41b).

$$(41) \quad a) \quad H = \tau - (\alpha/\beta w)^\sigma (M - \mu) ;$$

$$b) \quad M = \mu + (\beta w/\alpha)^\sigma (\tau - H) .$$

As in the Stone-Geary case, the expansion paths for different wage rates are rays emanating from the point $(M=\mu, H=\tau)$. The divergence of the implied expansion paths for any two wage rates will be greater, the higher is the elasticity of substitution, σ .

Expansion-path diagrams for modified C.E.S. functions with $\sigma = 0.5$ and 1.5 are shown in Figures 6 and 7. The values of μ and τ are the same as in the Stone-Geary illustration of Figure 5, and the ratio (β/α) is chosen to yield the same expansion path at $w = \$4/\text{hr.}$ in all cases. In all C.E.S. functions the income effect becomes stronger at low wage rates, and the strength of this interaction is an increasing function of σ . In the illustrated case with $\sigma = 1.5$, for instance, the income effect of $-.0625$ at $\$4.00$ per hour increases in magnitude to $-.4$ at $\$1.00$ per hour, a more-than-six-fold increase. The strength of this interaction implies a need for considerable caution in the use of labor supply functions based on utility functions with high substitution elasticities. In particular, for those cases where we wish to predict labor supply behavior in the face of income supplements and reduced marginal wages, we need to be sensitive to the fit of our estimated function over the region approaching those values.

Wage rates are implicit in the slopes of the budget lines in the expansion-path diagrams. The direct functional relationships between wage rates and work hours are illustrated in Figures 8a - 8c

Figure 6. Modified C.E.S. Utility Function $\sigma = 0.5$
Expansion Paths and Labor Supply Curves

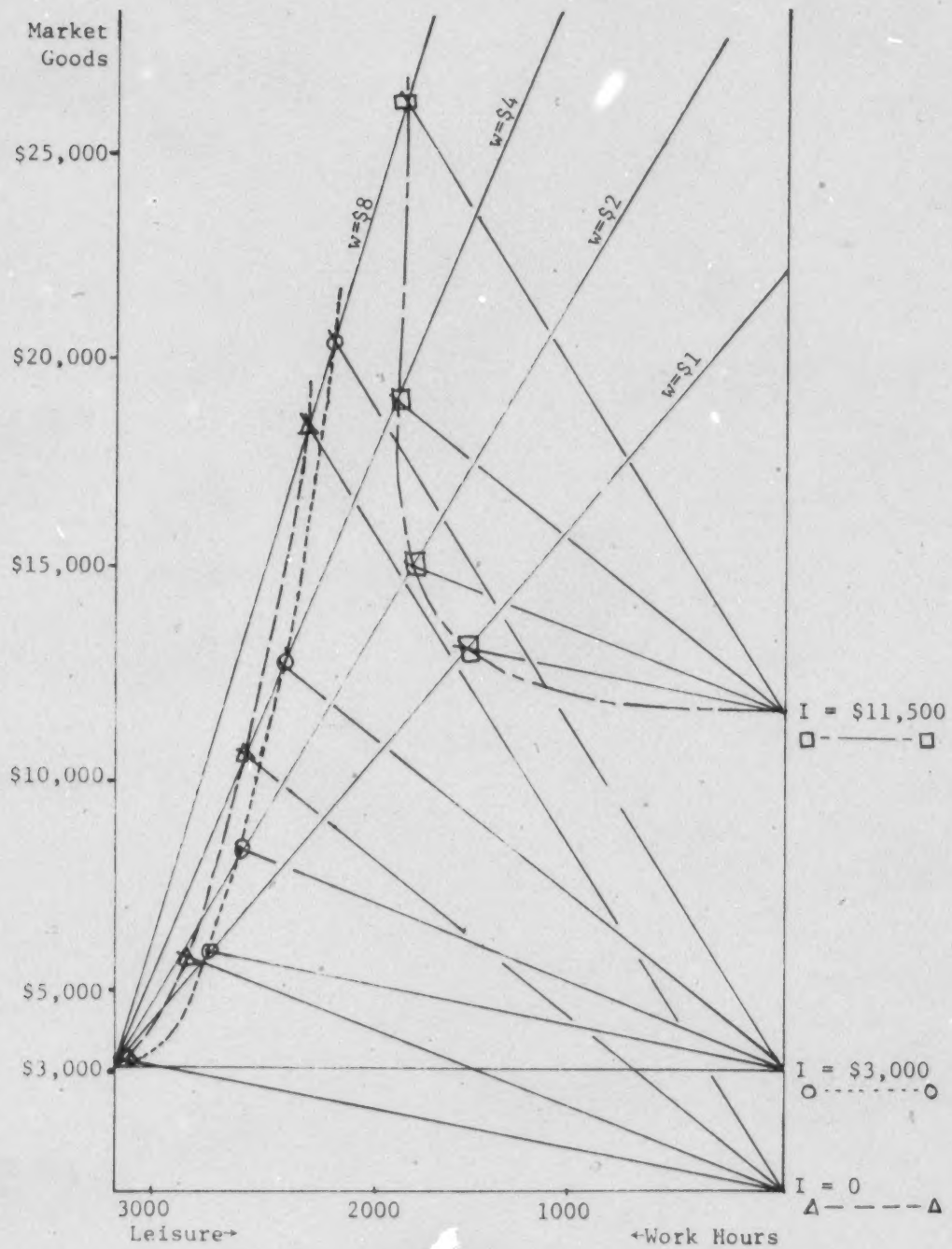
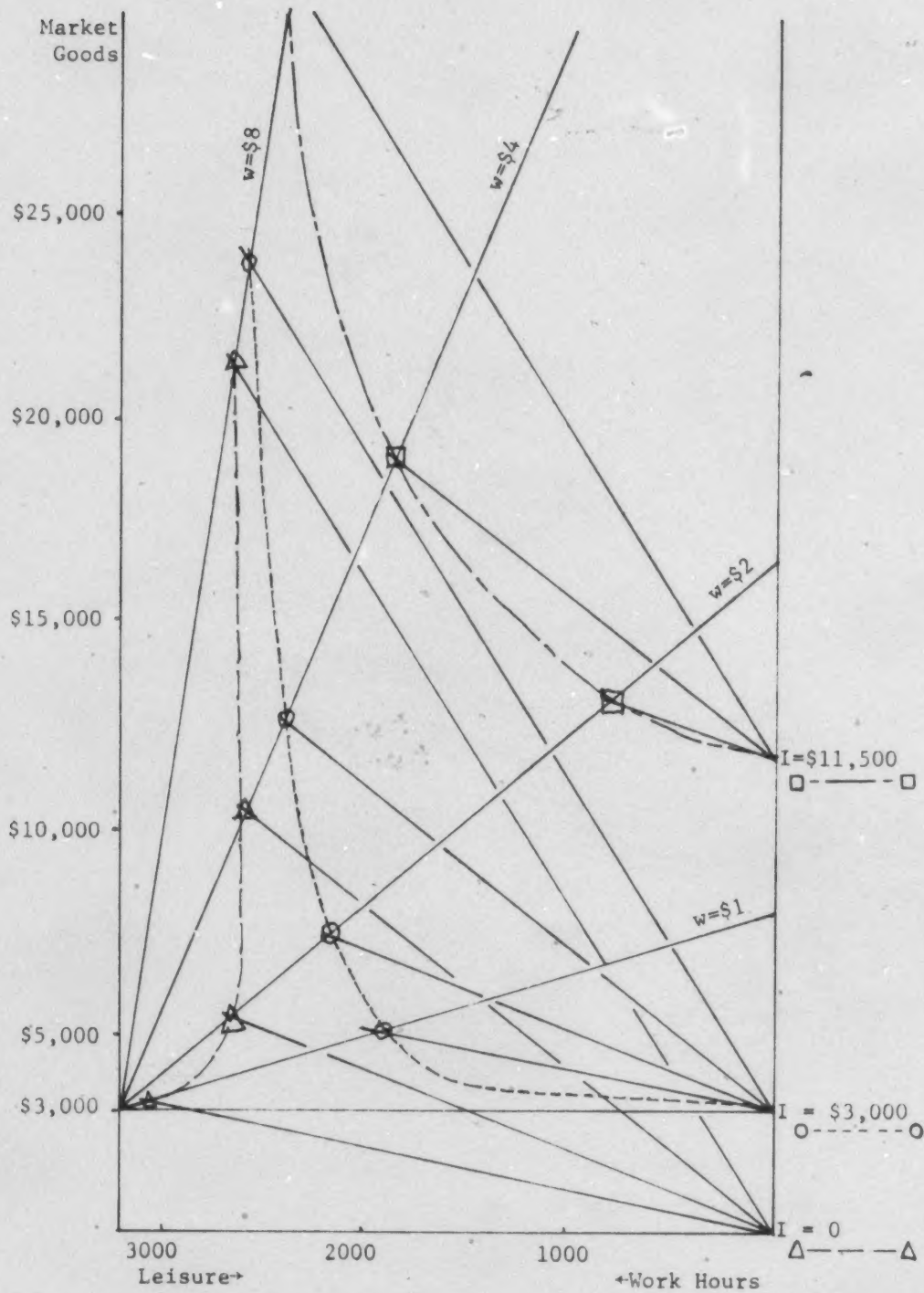


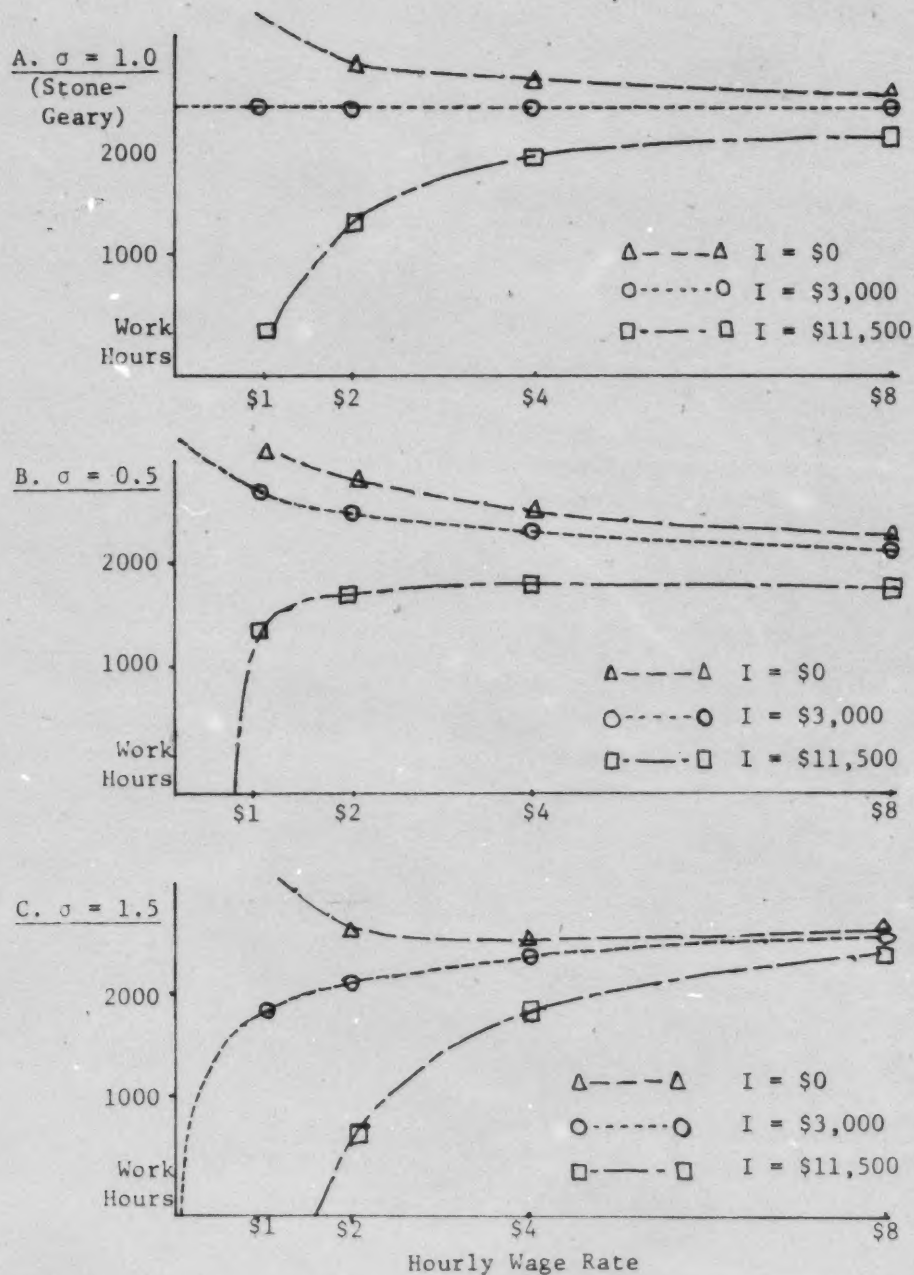
Figure 7. Modified C.E.S. Utility Function $\sigma = 1.5$
Expansion Paths and Labor Supply Curves



for the cases with $\sigma = 1.0, 0.5$, and 1.5 , respectively. The unit elasticity corresponds to the special Cobb-Douglas or Stone-Geary case. In all cases the labor supply responses to changes in the wage rate depend on the level of $(I - \mu)$ in a manner similar to the Stone-Geary case, though the direct correspondence between the direction of the response and the sign of $(I - \mu)$ holds only in that case. The differences in wage responses are most pronounced in the range of low wage rates, while in the limit of high wage rates all of the wage curves converge to the curve for $(I - \mu) = 0$. For elasticities greater than unity, that curve is positively sloped and approaches the constant asymptote, $H = \tau$. For smaller elasticities, the curve for $(I - \mu) = 0$ has a negative slope and approaches a zero asymptote. In general, a higher elasticity results in a positively sloped supply curve over a broader range of wage rates for any given value of $(I - \mu)$.

Wales initially estimated the subsistence parameters as well as the substitution elasticity and the ratio (α/β) . He deemed those results to be unsatisfactory and did not report them, because "many" sample individuals were observed to be in positions below the estimated minimum acceptable levels of leisure or income. After constraining the subsistence parameters to the values $\mu = \$6,500$, $\tau = 4160$ hours (our notation), he obtained estimates of σ ranging from 0.76 to 2.08 . His preferred estimates for the full sample of businessmen are $\hat{\sigma} = 1.51$ and $(\alpha/\beta) = 1.54$. These parameter values are candidates for construction of a null hypothesis in tests of various functional forms.

Figure 8. Labor Supply Curves as Direct Functions of the Wage Rate at Different Levels of Nonwage Income:
Three C.E.S. Functions: $\sigma=1.0, 0.5, 1.5$



Utility Functions with Variable Elasticity of Substitution

A labor supply model based on a homothetic utility function more general than the C.E.S. function is comparatively easy to derive in the single-worker case. In this model we do not obtain an explicit function for utility but rather work from an expression for the elasticity of substitution, given in (42). The function $g(w)$ may be virtually any continuous and differentiable function, and the particular form of (42) is chosen for subsequent ease of integration.

$$(42) \quad \sigma(w) = -w g'(w) .$$

In the general production function, the elasticity of substitution is defined as the elasticity of factor proportions with respect to the factor-price ratio. In the present application, the arguments of the function are leisure and market goods less respective subsistence requirements, and the price ratio is simply the wage rate, because we have already defined w relative to the price of market goods. The elasticity of substitution is thus expressed as shown in equation (43).

$$(43) \quad \sigma(w) = -wg'(w) = - \frac{\frac{L - \lambda}{M - \mu}}{\frac{\partial}{\partial w}} \cdot \frac{w}{M - \mu} .$$

The function in (43) may be rearranged and integrated to yield an expression for the expansion paths corresponding to the variable elasticity of substitution implied by a particular choice of $g(w)$.

$$(44) \quad a) \quad \frac{\frac{L - \lambda}{M - \mu}}{\frac{L - \lambda}{M - \mu}} = \frac{wg'(w)}{w} \frac{dw}{w};$$

$$b) \quad \log \frac{L - \lambda}{M - \mu} = g(w) + \log k;$$

$$c) \quad \frac{L - \lambda}{M - \mu} = k e^{g(w)}.$$

Substitutions from the labor-leisure identity then yield the expressions for the expansion paths in terms of work hours and market goods. The two expressions in (45a) and (45b) are simple reciprocals.

$$(45) \quad a) \quad H = \tau - k e^{g(w)} (M - \mu);$$

$$b) \quad M = \mu + k^{-1} e^{-g(w)} (\tau - H).$$

Finally, the labor supply functions are derived by simultaneous solution of the expansion-path function and the budget constraint. This solution is the analytic equivalent of the intersections of budget lines and expansion paths plotted in Figures 5-7. The resulting supply functions are given by (46) for any desired $g(w)$.

$$(46) \quad H = \frac{\tau - k e^{g(w)} (I - \mu)}{1 + k w e^{g(w)}}.$$

The function $g(w) = -\sigma \log w$ yields the supply function for the C.E.S. cases. Any $g(w)$ that is a linear function of powers of w adapts simply to the variable elasticity model, and other functions may presumably be adapted with varying degrees of ease. The implied supply functions will of course be correspondingly complex.

At the present time this author is unaware of any labor supply studies that use a variable elasticity model. Samuel Rea's general estimates of labor supply parameters, which were discussed in section II, suggest that such a model might be appropriate, however.

Quadratic Utility Function

The quadratic utility function has been used as the basis for a number of studies of commodity demand and for illustrative purposes in the context of labor supply.²⁹ A general form of this function for leisure and market consumption of a single-worker family may be expressed as shown in (47).

$$(47) \quad U = a + bL + cL^2 + dLM + eM + fM^2 .$$

The coefficients b and e are positive, while c and f are negative, and d may be of either sign within the limits set by the second-order conditions.

The utility function has a maximum or "bliss" point at the values of leisure and market goods given in (48).

$$(48) \quad a) \quad L^* = \frac{-2bf + de}{4cf - d^2} ;$$

$$b) \quad M^* = \frac{-2ce + db}{4cf - d^2} .$$

The indifference curves implied by the quadratic utility function are concentric about the bliss point. This function will thus be a useful approximation only if that point falls well above the observed ranges of leisure and market goods. The expansion

paths converge to the bliss point with increasing income and are described by the expressions in (49), with T again representing total available time.

$$(49) \quad a) \quad H = \frac{T - we - b + (2wf-d)M}{2c - wd};$$

$$b) \quad M = \frac{b - we + (2c - wd)(T-H)}{2wf - d}.$$

The convergence of expansion paths at different wage rates is indicated by the slope of the paths.

$$(50) \quad \frac{\partial H}{\partial M} = \frac{-(2wf - d)}{(2c - wd)}.$$

If the cross-product parameter, d , is zero, the slope is simply proportional to the wage rate. For acceptable negative values of d , the magnitude of the slope increases more than proportionately with increasing wage rate; conversely, for positive values of d , the slope increases less than proportionately with increasing wage rate.

The model based on a quadratic utility function is thus similar to models that assume a linear additive income effect in the supply equation, in that both imply convergent expansion paths. The pattern of convergence is more complex in the present case and is generally stronger in the lower range of wage rates. The convergence property does not have a high degree of intuitive theoretical appeal. In the linear additive case, it follows from a model with the compensating virtue of a simple and empirically tractable supply function. That virtue is notably lacking in the quadratic utility model, as is evident from the form of the supply function given in (51).

$$(51) \quad H = \frac{-(dT + e)w - 2(fw - d)I + 2cT + b}{2(fw^2 - dw - c)}.$$

We will thus incline toward use of this function in empirical work only if the estimates from the general interactive function strongly suggest that it is the appropriate form.

Parallel Utility Functions

In the preceding discussion we have considered two general types of utility functions. Homothetic functions are characterized by expansion paths that diverge from a minimal subsistence point of leisure and market goods, while concentric functions are characterized by expansion paths that converge at a bliss point in the upper range of the basic goods. As we have noted, the concentric functions do not have a great deal of intuitive appeal but may prove to be useful approximations over some ranges of variables.

Homothetic functions have received a great deal of attention in literature on demand and production because they imply conveniently regular behavior patterns when variables are expressed in proportional terms. For instance, any given indifference curve is identical, but for a factor of scale, to any other curve for the same function. In the labor supply context, however, many of the convenient features of the homothetic models are of little avail. Leisure is essentially only a construct of the model and has no obvious and reasonable zero point.³⁰ Proportional responses in leisure demand are thus critically dependent on the choice of the subsistence point. Furthermore,

labor supply, which is of primary concern in the model, is defined as the additive complement of leisure, so even the arbitrary regularity of proportional leisure-demand responses does not carry over to labor supply response.

The additive complementarity of the model of labor supply and leisure demand suggests a natural third class of utility functions, those that are homothetic in an absolute rather than a proportional sense. Functions in this class are characterized by parallel expansion paths and identically shaped parallel indifference surfaces, hence our terminology of "parallel utility functions."

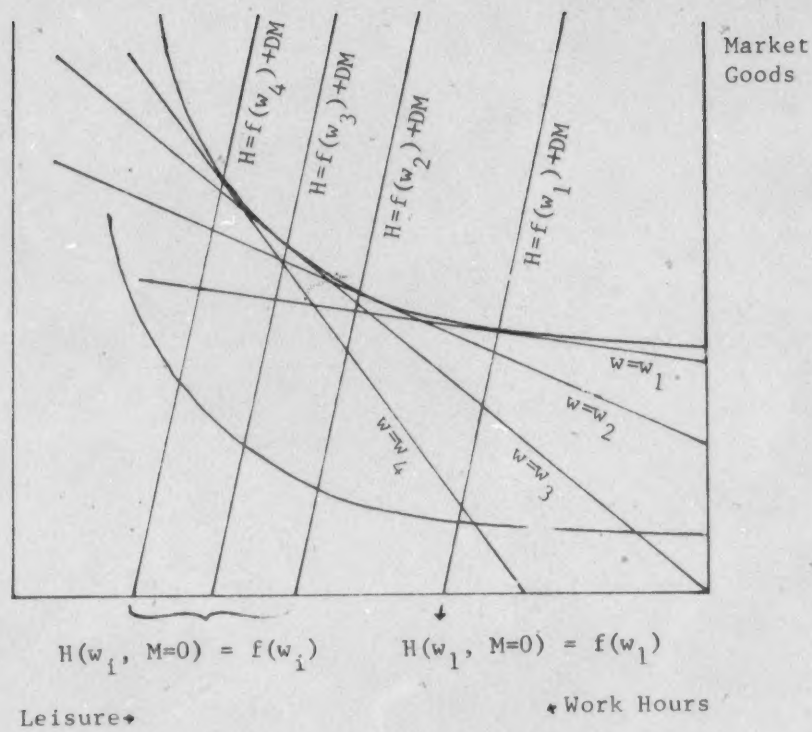
Parallel utility functions do not, in general, lend themselves to explicit analytic expressions of the utility function, though such expressions may be derived in some cases.³¹ The approach we follow here is to describe the utility functions implicitly through the expression for the corresponding sets of expansion paths. Given that parallel expansion paths have equal slope, which we denote by the negative constant, $D = \left. \frac{\partial H}{\partial M} \right|_w$, the expansion paths may be expressed in the form shown by equation (52).³²

$$(52) \quad H = f(w) + DM.$$

The function $f(w)$ determines the "spacing" of the expansion paths and thus determines the functional form of the substitution effect implied by the model. The graphical illustration of a parallel utility function, shown in Figure 9, helps to clarify the role of the "spacing" function $f(w)$.

We may use the analytical equivalent of the graphical technique demonstrated in Figures 5-7 to derive the labor supply functions.

Figure 9. Expansion Paths and Indifference Curves for a Parallel Utility Function



That is, we simply solve equation (52) simultaneously with the budget constraint. Substitution for M in (52) from the budget constraint gives

$$(53) \quad H = f(w) + D (wH + I) ,$$

which yields the general solution for the supply function.

$$(54) \quad H = \frac{f(w) + D I}{1 - D w} .$$

A number of general features of this model may be seen in the expressions for the various partial derivatives.

$$(55) \quad \frac{\partial H}{\partial I} = \frac{D}{1 - Dw} .$$

$$(56) \quad \frac{\partial H}{\partial w} = \frac{(1 - Dw) f'(w) + D (f(w) + DI)}{(1 - Dw)^2} = \frac{f'(w) + D H}{1 - Dw} .$$

$$(57) \quad \left. \frac{\partial H}{\partial w} \right|_u = \frac{\partial H}{\partial w} - H \frac{\partial H}{\partial I} = \frac{f'(w)}{1 - Dw} .$$

The form of the interaction between the income effect and the wage rate, as shown in (55), depends only on the parameter D . The magnitude of the income effect declines as the wage rate increases, as in the case of homothetic functions, but the interaction is weaker in the present case.³³

The wage slope, given in (56), may be either positive or negative. The denominator is always positive, and the two terms in the numerator have opposite signs. The constant D is negative, so the term DH is negative and $f'(w)$ must be positive to provide a positive substitution effect. The relative strengths of the two terms in the numerator will determine whether the individual labor supply function is rising or falling.

The substitution effect, shown in (57), depends on $f'(w)$ and on the parameter D but is independent of the level of nonwage income. We may thus denote it as a function of the wage rate alone. Intuitively, the simple form of (58) is a consequence of the identical shapes of all indifference curves for a given parallel function.

$$(58) \quad \left. \frac{\partial H}{\partial w} \right|_u = S(w) .$$

The relationships (55)-(57) may be used to investigate the properties of various functions in this class. In the following discussion we consider three variants of parallel utility functions that may have useful properties.

Case 1: The Ashenfelter-Heckman model. A labor supply function with a constant substitution effect has been suggested by Ashenfelter and Heckman (1973, 1974). As was shown in section II, this property is incompatible with the constant income effect that they also intended to incorporate into their model. A reasonable interpretation of their estimation procedure suggests that they in fact estimate a labor supply function that has neither of the assumed properties but is consistent with a utility function in the parallel class.

Interpretation of the underlying Ashenfelter-Heckman model is not entirely straightforward because their discussion of the systematic variation of labor supply is intertwined with their discussion of the random component. However, if we consider a worker whose labor supply is determined by an exact function, the form of the function implied by the Ashenfelter-Heckman estimation model appears to be

$$(59) \quad H = A + S^*w + B^*M ,$$

where we employ our notation.

The expression in (59) may be interpreted as a special case of the expansion-path expression (52), with D equal to B^* and the function $f(w)$ given by (60).

$$(60) \quad f(w) = A + S^*w .$$

The expansion-path expression (59) may not be used directly for ordinary least squares estimation because M is an endogenous variable. To circumvent this difficulty, Ashenfelter and Heckman employ an instrumental variable for M in the estimation of the model.³⁴

In our alternative procedure, we solve for a reduced form of the supply function by substitution for M from the budget constraint.

$$(61) \quad a) \quad H = A + S^*w + B^*(wH + I) ;$$

$$b) \quad H = \frac{A + S^*w + B^*I}{1 - B^*w} .$$

The expression in (61b) is again a special case of the general parallel supply function (54).

The income and substitution effects implied by this model follow immediately from (55), (57), and (60).

$$(62) \quad \frac{\partial H}{\partial I} = \frac{B^*}{1 - B^*w} .$$

$$(63) \quad S(w) = \frac{f'(w)}{1 - B^*w} = \frac{S^*}{1 - B^*w}.$$

Both the income effect and the substitution effect thus are not constants but are functionally dependent on the wage rate except in the degenerate case when B^* is equal to zero.

Ashenfelter and Heckman obtain estimates of S^* in the range from 66.9 to 68.9 annual hours per dollar of hourly wage and estimates for B^* that range from $-.065$ to $-.070$ hours per dollar of market consumption. An expansion-path diagram illustrating the implications of these estimates is shown in Figure 10. For the figure we have used the round values $S^* = 70$ and $B^* = .065$ and chosen the constant A to imply an equilibrium of 2000 hours for a worker with no outside income and a net wage of \$4.00 per hour. The model clearly shares one of the weaknesses that affected a number of the classical general models: The implied zero-wage equilibrium exceeds 1500 hours per year for incomes up to \$11,000. The model is well-behaved in the high-wage range, however.

Case 2: A constant substitution effect in a function of the parallel class. It is straightforward to derive a supply function in the parallel class that has the property of a constant substitution effect. The substitution effect in (57) is simply set equal to a constant, S , and the differential equation is solved for the function $f(w)$.

$$(64) \quad a) \quad f'(w) = (1 - Dw) S ;$$

$$b) \quad f(w) = A + Sw - \frac{D}{2} Sw^2 .$$

The supply function then follows directly from (54):

$$(65) \quad H = \frac{A + Sw - 1/2 DSw^2 + DI}{1 - Dw} .$$

Differentiating with respect to the wage rate and simplifying then yields an expression for the uncompensated wage effect that is exactly in the classical Slutsky form.

$$(66) \quad \frac{\partial H}{\partial w} = S + \frac{D}{1 - Dw} \cdot H .$$

The supply curve will be either rising or falling depending on the relative sizes of the income and substitution effects in (66). For similar values of parameters, the implications of this model will not be very different from those of the Ashenfelter and Heckman model. The indifference curves are true parabolas, which imply zero marginal valuation of leisure at a level of work hours not far below standard full-time work.³⁵

Case 3: Parallel function with a proportional substitution effect. The two variants of the parallel model discussed above have the disadvantage that their zero-wage expansion paths may fall in a range not far below full-time work effort. This implies a zero or negative valuation of any leisure gained by further reduction of work effort. A simple function that avoids this property makes the substitution effect inversely proportional to the wage rate.

$$(67) \quad S(w) = \gamma/w .$$

Solution of (57) yields the expression for the function $f(w)$

$$(68) \quad f(w) = A + \gamma \cdot \log w - \gamma Dw ,$$

which leads to the supply function by way of (54).

$$(69) \quad H = \frac{A + \gamma \log w - \gamma Dw + DI}{1 - Dw} .$$

This form of supply function implies steeply rising curves at low wage rates, which bend backward to gradually falling curves at high wage rates as the substitution effect gets smaller. The expansion-path diagram would be similar to that shown in Figure 9, though no precise functional form was employed there.

Clearly a virtually unlimited number of functional forms could be adapted to the parallel utility function, but there is no great benefit to our further investigation of other variants at this juncture. The examples presented above establish parallel functions as a flexible and well-behaved class of preference structures that may prove to be appropriate as an analytical basis for models of labor supply.

IV. Conclusion

In this paper we have sought to provide the theoretical background for the explicit modeling and estimation of income-leisure preference structures. Estimates of preference structures have been implicit or explicit in previously estimated models of labor supply, but in almost all cases those estimates have been limited by arbitrary assumptions about functional form. Our intent in this study has been to broaden the scope of model construction so that not only the estimation of individual parameters but also the choice of an appropriate functional form may be viewed as a tractable empirical question.

One of the reasons that the selection of an appropriate functional form has not been regarded as an empirical question in previous labor supply studies is that a systematic set of alternatives to choose among has not been obvious. In this paper we have characterized the inherent preference structures of a wide variety of labor supply models in terms of patterns of expansion paths in the income-leisure plane. This methodology provides the common ground for a systematic comparison of the underlying preference structures from a highly diverse set of analytical forms. In particular, the preference structures implicit in various functional adaptations of the classical general labor supply model may be compared directly with those from models based on explicit assumptions about the form of the utility function.

A large part of the paper has been devoted to a review of empirical labor supply models and an exposition of their underlying preference structures within the expansion-path format. On the basis of this review it has been possible to outline a well-behaved set of alternative structures. The primary distinguishing feature of the different models is the convergence or divergence of expansion paths at different marginal wage rates. The structures implied by the more tractable empirical models fall along a continuum from highly divergent to highly convergent patterns of expansion paths. At the divergent end of the spectrum are models based on homothetic utility functions, such as the model with constant elasticity of substitution or the more general model with variable elasticity. In these models, the expansion paths diverge from a subsistence point representing minimum acceptable levels of leisure and market

consumption. At the other end of the continuum are preference structures with expansion paths that converge at a "bliss point," so named because it represents a maximum attainable level of utility. The linear additive supply model and the quadratic utility model are the primary examples of convergent preference structures.

A third general class of preference structures falls midway between the two extremes. Structures in this class are characterized by parallel expansion paths, hence our "parallel preference functions." To our knowledge, this class of models has not previously been formulated in analytical terms, although a special case of a parallel preference model was implicit in an estimation procedure employed by Ashenfelter and Heckman. As examples, we derived the analytical forms for two other simple parallel models.

The expansion-path methodology that served to organize our review of diverse analytical preference structures also serves as the basis for a general empirical model. Our proposed estimation form is adapted from a model developed by Cohen, Rea, and Lerman (1970). In this model, categorical interactions between the marginal wage rate and the income effect permit separate estimates of the income effect for each of several intervals of the marginal wage rate. These estimates may then be interpreted as approximate sample analogs of expansion paths for the various wage intervals. The direction and magnitude of estimated interactions between the income effect and the marginal wage rate will determine the convergence or divergence of the preference structure implied by the general estimation model. If the magnitude of the estimated

income effect had a strong inverse relationship with the marginal wage rate, the general estimates would suggest a divergent homothetic structure. A weak inverse relationship would be consistent with a parallel structure, and a zero or positive relationship would imply a convergent structure.

It is also possible to conduct formal statistical tests of alternative functional forms on the basis of estimates from the general empirical model. To test the appropriateness of a given analytical form, we would establish a null hypothesis involving the implied ratios of income effects at different wage levels. We would then reject, or fail to reject, the null hypothesis on the basis of freely estimated income effects for the different wage levels.

In any empirical study one must expect some degree of uncertainty due to sampling errors in estimates. It is thus likely that estimates from the general model will be statistically consistent with a number of different analytical forms within the range of available data. The different functional forms can have very different implications about behavior at more extreme values of independent variables, however. These differences are important because policy issues, such as the assessment of the labor supply impact of income-maintenance programs, often involve inferences about responses outside the usually observed range of variables. In our review of alternative models we have thus devoted additional attention to the general plausibility of implied labor supply behavior over a wide range of variables. The differing implications of different models would be

testable, given sufficiently rich empirical data. In the absence of reliable empirical information, our review provides useful perspective on the sensitivity of policy inferences to alternative assumptions about functional form.

In the above discussion we have emphasized the generality of our proposed approach to empirical inference about the functional form of preference structures. We also need to note its limitations. In particular, the well-behaved set of alternative preference structures, characterized by divergent, parallel, or convergent expansion paths, is limited to models having linear expansion paths. A more general set, including models with curved expansion paths, would clearly be more difficult to characterize in simple terms. It is our judgment that our restriction to models with linear expansion paths represents a reasonable compromise given the limitations of available data on individual labor supply. Most critically, the variance of nonwage income is rarely large in observed data, and the effective variance is further reduced because separate income effects must be estimated for individuals at different wage levels. It is thus unlikely that workably reliable estimates of curvature in the income effects can be obtained in addition to estimates of wage-income interactions. We have included an exposition of selected properties of previously estimated curvilinear supply models but have not attempted a more general treatment of curvilinear structural models. These limitations notwithstanding, we hope that the methods proposed in this paper can lead to significantly greater generality in the estimation and interpretation of models of labor supply. There are also potential extensions of this

methodology to multiple-worker models of labor supply and to models of demand for major consumer commodities.

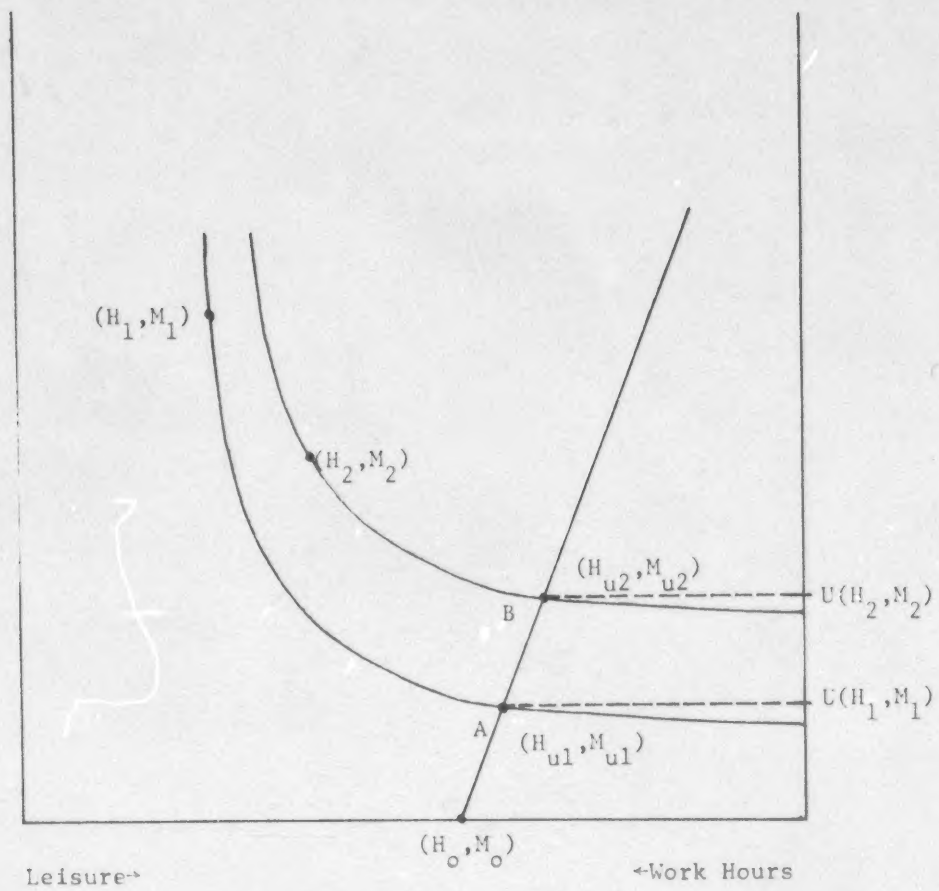
Appendix.Explicit Utility Functions for Members of the Parallel Class

For some functions in the parallel class, it is possible to solve for explicit expressions for utility as a function of market consumption and leisure or work hours. Since our primary interest is in work hours, we shall work directly with that negative good, rather than with its positive complement, leisure, as an argument of the utility function. The basic method of solution is illustrated graphically in Figure A.1.

Our aim is to find a monotonic index that will provide a relative ranking of the utility level of all consumption points in the H, M plane. We accomplish this by obtaining an equation for the indifference curve through any given consumption point and then determining the point of intersection between that curve and an expansion path that has been chosen as a reference.³⁶ Higher indifference curves will clearly intersect the reference path at a higher level of market consumption, and conversely for lower curves, so the level of market goods at the intersection point provides a convenient index of utility.³⁷ In the figure, the indifference curve through the consumption point (H_1, M_1) intersects the reference path at point A, with coordinates (H_{u1}, M_{u1}) . The index value for utility at that point is then taken to be $U(H_1, M_1) = M_{u1}$. The value of the utility index for consumption point (H_2, M_2) is obtained in analogous fashion.

In the analytical derivation that follows, the notation corresponds to that used in Figure A.1. We also use w , the notation for

Figure A.1



wage rate, to represent the marginal wage rate of substitution at any consumption point. Since one of the objects of the utility index is to allow comparison of consumption points that are not necessarily local optima, the marginal rate of substitution or the equilibrium marginal wage rate is presumed to be initially unknown. For convenience of reference, a summary of notation follows:

(H_i, M_i)	The consumption point of work hours and market goods for which we wish to obtain a value of the utility index.
w_i	The (initially unknown) marginal rate of substitution or equilibrium marginal wage rate at (H_i, M_i) .
(H_{ui}, M_{ui})	The (initially unknown) point of intersection between the reference expansion path and the indifference curve through (H_i, M_i) .
$U_i = M_{ui}$	The value of the utility index.
w_o	The marginal rate of substitution along the reference path--a known constant chosen for convenience.
(H_o, M_o)	The origin point of the reference path. M_o is most conveniently taken to be zero, and H_o is a constant that depends on w_o . H_o , M_o , and w_o are all implicit or explicit constants in the function $f(w)$ that characterizes the parallel function in question.

The features of the parallel function that are presumed to be known are the general expression for the expansion paths and the derived expression for the substitution effect, shown in (A.1) and (A.2). These equations are repeated from (52) and (57), respectively.

$$(A.1) \quad H = f(w) + DM.$$

$$(A.2) \quad S(w) = \frac{f'(w)}{1 - DW}.$$

In the particular cases treated below, we work directly from alternative assumed forms of the function $S(w)$ and do not bother with the expressions for $f(w)$, which were derived in section III.

The equation for an indifference curve is obtained as the solution to two simple simultaneous differential equations based on the form of the substitution effect and the definition of the marginal wage rate. We solve for these equations as if the initial constants (H_{u1}, M_{u1}) were known. We then solve for the value of the initial constants that will yield an indifference curve through the consumption point (H_1, M_1) .

The function for the substitution effect relates work hours to the wage rate along an indifference curve.

$$(A.3) \quad a) \quad S(w) = \left. \frac{\partial H}{\partial w} \right|_u ;$$

$$b) \quad \partial H_u = S(w) \partial w_u .$$

The definition of wage rate relates all three variables, market consumption, work hours, and wage rate.

$$(A.4) \quad a) \quad w = \frac{\partial M}{\partial H} ;$$

$$b) \quad \partial M = w \partial H .$$

We may reduce the number of variables in (A.4) by substitution for ∂H from (A.3b).

$$(A.5) \quad \partial M_u = w S(w) \partial w_u .$$

Given knowledge of the form of $S(w)$, the equations (A.3b) and (A.5) may be solved by integration, and those solution functions may then be solved for the desired initial constants.

The simplest functional form for the substitution effect is that it is constant.

$$(A.6) \quad S(w) = S.$$

The differential equations then become

$$(A.7) \quad \partial H_u = S \partial w_u ;$$

$$(A.8) \quad \partial M_u = w S \partial w_u .$$

These admit to the integral solutions in (A.9) and (A.10), with w_o conveniently chosen to be zero.

$$(A.9) \quad a) \quad H \Big|_{w_o}^{w_i} = \int_{w_o}^{w_i} S \partial w ;$$

$$b) \quad (H_i - H_{ui}) = S(w_i - w_o) = S w_i .$$

$$(A.10) \quad a) \quad M \Big|_{w_o}^{w_i} = \int_{w_o}^{w_i} w S \partial w ;$$

$$b) \quad (M_i - M_{ui}) = \frac{1}{2} S (w_i^2 - w_o^2) = \frac{1}{2} S w_i^2 .$$

The unknown, w_1 , may be eliminated by simultaneous solution.

$$(A.11) \quad (M_1 - M_{ui}) = \frac{1}{2S} (H_1 - H_{ui})^2 .$$

Then the expression for the reference expansion path given in (A.12) is used to reduce equation (A.11) to a quadratic equation in the single unknown, M_{ui} , which is our utility index.

$$(A.12) \quad H_{ui} = H_o + D (M_{ui} - M_o) = H_o + DM_{ui} .$$

$$(A.13) \quad (M_1 - M_{ui}) = \frac{1}{2S} (H_1 - H_o - DM_{ui})^2 .$$

Rearranging and applying the quadratic formula to (A.13) finally yields the explicit function for our utility index.

$$(A.14) \quad U(H_1, M_1) = M_{ui} = \frac{1}{D^2} \left[-S + D(H_1 - H_o) + \sqrt{S^2 - 2SD(H_1 - H_o - 2DM_1)} \right] .$$

For reassurance that this is indeed the appropriate utility function, it is comparatively easy to set the ratio of marginal utilities equal to the negative wage rate and derive the expansion-path expressions that are more directly derived in equations (52) and (64b).

The parallel utility function with a proportional substitution effect, $S(w) = \frac{Y}{w}$, turns out to be less tractable in terms of an explicit utility index. A derivation parallel to that above leads to the following equation, with M_{ui} as an unknown.

$$(A.15) \quad (M_1 - M_{ui}) = -w_o [1 - \exp (H_1 - H_o - DM_{ui})] .$$

So far as I am aware, equation (A.15) does not admit to an analytic solution for M_{ui} .

Notes

¹We shall refer to preference structures and utility functions interchangeably. The former term is somewhat preferable in that it connotes only relative preferences. All monotonic transformations of a given utility function will correspond to a single preference structure. We will be concerned only with the relative preferences that are independent of the chosen scale of the utility function. We also use "income-leisure" as a shorthand for "market consumption-leisure" in reference to the arguments of the preference function. In other contexts we take care to reserve the term "income" to refer to exogenous nonwage income, as distinct from "market consumption" or "total family income," which is an endogenous variable.

²Dickinson (1975a).

³These are important issues for any practical applications of labor supply models and are addressed in Dickinson (1975a, 1976b).

⁴The use of a single composite of market goods with a single price index is a reasonable approximation to the extent that price changes in major subcomponents are approximately proportional or that those components with disproportionate price changes are not important substitutes for or complements to leisure.

⁵In this discussion we review and illustrate the properties of the general theoretical model. For a formal derivation of the classical restrictions see, for instance, Kosters (1966).

⁶In conventional demand studies the several demand functions in a complete set, such as the functions in (3a) and (3b), are often

treated symmetrically. Consistency with the budget constraint is then enforced by the imposition of so-called aggregation conditions. These conditions are discussed in Goldberger (1967). By incorporating the budget constraint in the demand equation for market goods we assure that the aggregation conditions will be met at all points for which the functions are defined.

⁷ Except for the negligible second-order effect due to the change in the marginal wage rate and the resulting response.

⁸ Recall that the signs are reversed from the usual demand theory results.

⁹ A change in only the marginal wage is equivalent to a change in wage rate for all hours accompanied by a compensating change in nonwage income equal to $-H_0 \Delta w$. This may be seen by extrapolating the line CK back to point P at zero work hours. The vertical distance AP represents the compensating income change, $-H_0 \Delta w$.

¹⁰ Garfinkel (1974) made direct use of estimated supply functions in projecting the labor supply responses implied by a variety of previously estimated supply functions. At that time, however, he appeared to regard the procedure as a second best, adopted only for expositional simplicity. Garfinkel and Masters (forthcoming) and Plotnick and Skidmore (1975) have adopted the direct simulation technique.

¹¹ It is necessary to use intermediate estimates of the substitution effects for more complex problems such as the determination of the expected responses by workers above the breakeven point of a negative income tax. Care must nonetheless be exercised to maintain

functional consistency. For direct use of the supply function as the simulation equation, all budget variables must be expressed in net terms. The necessary transformations are discussed in Dickinson (1975b).

¹²A partial listing includes Kosters (1966), Greenberg and Kosters (1970), Boskin (1973), Hill (1973), Dickinson (1974). In several cases other more complex forms were also estimated.

¹³See Cain and Watts (1973, p. 333, Table 9.1, footnote*).

¹⁴See Dickinson (1974) and in the context of other models, Ashenfelter and Heckman (1973) and Rea (1971).

¹⁵See, for instance, Greenberg and Kosters (1970), Rosen and Welch (1971), and Hill (1973). These authors note the dependence of the elasticity on the initial wage but then treat it as a constant.

¹⁶ $S = A - BH$, and B is negative.

¹⁷ $\left. \frac{\partial H}{\partial w} \right|_u \cdot \frac{w}{H} = \eta_s$, a constant less than one. A given proportional increase in w results in a smaller proportional increase in H . The ratio w/H thus increases with increasing w and H along an indifference curve, and $\left. \frac{\partial H}{\partial w} \right|_u$ must decrease to maintain the assumed constancy of the elasticity. Conversely, as w approaches zero, $\left. \frac{\partial H}{\partial w} \right|_u$ must become very large.

¹⁸I maintain this somewhat awkward terminology to avoid confusion with the "elasticity of substitution" parameter for production functions, which is similar but not identical to the current concept.

¹⁹The uncompensated wage derivative, $\frac{\partial H}{\partial w}$, must be independent of I , since $\frac{\partial H}{\partial I} = B$ is a constant and is thus independent of w . $\partial \left(\frac{\partial H}{\partial I} \right) / \partial w = 0$ implies that $\partial \left(\frac{\partial H}{\partial w} \right) / \partial I = 0$.

²⁰This constraint has the greatest effect on the income effect for the wage interval \$2.50-\$4.99/hr. A free approximation of the expansion path for that range would intersect the next-lower path in the vicinity of $H=2250$, $M=\$8500$.

²¹An extensive discussion of our treatment of complex budget parameters is provided in Dickinson (1975b). See also the brief comments later in the text.

²²For an application of this functional form in a labor supply context, see Wales and Woodland (1974). For a more general exposition and application, see Christensen, Jorgensen, and Lau (1975).

²³See, for instance, Leuthold (1968), Wales (1973), and Horner (1973).

²⁴The parameters α and β clearly can be multiplied by a scale constant without affecting the supply functions. The functions are thus simplified slightly by normalization such that $\alpha + \beta = 1$. Note that the terms in 35c and 35d would be grouped differently for estimation purposes.

²⁵Note that in this particular case the slope of an expansion path, $\frac{\partial H}{\partial M} = \frac{-\alpha}{\beta w}$, is proportional to the income effect on work hours from (35c), $\frac{\partial H}{\partial I} = -\alpha/w$. This is a consequence of the linear expenditure property, which implies a constant marginal propensity to spend on leisure regardless of the wage rate. From equation (21) we have

$$\frac{\partial H}{\partial M} = \frac{\frac{\partial H}{\partial I}}{1 + w \frac{\partial H}{\partial I}} = \frac{-\alpha/w}{1 - w \alpha/w} = \frac{-\alpha/w}{(1-\alpha)} = \frac{-\alpha/w}{\beta} = \frac{1}{\beta} \frac{\partial H}{\partial I}.$$

The term $w \frac{\partial H}{\partial I}$ in the denominator reduces to a constant only for this particular functional form.

²⁶These parameter values were chosen with an eye toward a degree of realism for male family heads. The asymptote of 2400 hours is a bit high but is a conveniently round number. The labor supply curve for $I = 0$ is similar to that observed in several studies. And at \$4 per hour (close to the current mean wage) the income effect of $-.0625$ is well within the range of recent estimates.

²⁷Goldberger (1967), with reference to an unpublished paper by Pollak (1967), points out that the addilog from (37a) is simply a monotonic transformation of the more familiar form. The labor supply equations are unaffected by such transformations.

²⁸In our notation, σ is defined as a positive number.

²⁹See, for instance, Houthakker and Taylor (1970) and Taylor and Weiserbs (1972). Henderson and Quandt (1958, p. 24) present a simple labor supply example.

³⁰The literal zero point that would allow up to 24 hours of work per day clearly untenable implications for labor supply. Labor supply equilibria at normal levels would be possible only if the income effect were several times greater than has been observed in any empirical study.

³¹The derivation of an explicit utility index for one function in the parallel class is presented in the Appendix to this paper.

³²Note that the variable w in $f(w)$ refers to the equilibrium marginal wage rate or the marginal rate of substitution, which is fixed along a given expansion path. It is thus a characteristic of the preference structure and is not associated with a particular budget line. The marginal rate of substitution is equated to the wage rate of the budget constraint in equation (52) in order to solve for the labor supply equation.

³³The limiting case of a homothetic function with zero elasticity of substitution is also a member of the present class, with $f(w)$ equal to a constant. In that special case the wage-income interaction is, of course, the same.

³⁴Strictly speaking, Ashenfelter and Heckman use an instrumental variable for M itself only in the two-worker model (1974). In the single-worker model (1973) they construct the instrumental variable piecewise, using predicted values of work hours and other components. However, they average the individual predicted values of H with the constant sample mean in their construction of the actual variable. Had they used the sample mean alone, their model would be a variant of the linear additive supply function. Their use of an average value presumably results in an implicit preference structure that represents a mixture of the linear additive supply model and the parallel utility model.

³⁵The solution function for an indifference curve for this model is given in equation (A.11) of the Appendix.

³⁶This general approach to the specification of a utility function is related to one employed by Samuelson (1965, p. 785). In the case of a parallel function, the equation for any given indifference curve applies as well to all others with only a linear translation. Samuelson treats the homothetic case in which indifference contours are identical but for a scale factor. We use the translation parameter as the utility index; Samuelson uses the scale factor.

³⁷Any monotonic transformation of this index would also be acceptable as a utility index (see Samuelson, 1947, p. 94), but the present one seems to be the most convenient.

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